Solutions of SDE — uniqueness and locality

<u>last time:</u> $dX_t = a(t, X_t)dt + \sigma(t, X_t)dB_t$

strong solution: exists under uniform Lipschitz assumptions

<u>Q:</u>
- uniqueness ?
- locality
- relaxing uniform Lipschitz assumptions.

<u>Def</u> Assume standard setting, let T be a finite stopping time.

A <u>strong solution up to stopping time T</u> is stoch. process $\{X_t : t \in [0, T]\}$ s.t. $\{X_{T \wedge t} : t \geq 0\}$ is strong solution to SDE

$$dX_{T\wedge t} = 1_{T>t}\, a(t, X_{T\wedge t})dt + 1_{T>t}\, \sigma(t, X_{T\wedge t})dB_t$$

A strong solution up to (limit stopping) time $\tilde{T}$ is $\{\tilde{X}_t : t \in [0, \tilde{T})\}$ s.t. for some stopping times $\{T_n\}_{n\geq 0}$ with $T_n \uparrow \tilde{T}$, $\{X_{T_n \wedge t} : t \geq 0\}$ is a solution up to stopping time T_n.

Thm (Uniqueness and locality) Assume standard settings with coefficients $a(\cdot,\cdot), \sigma(\cdot,\cdot)$ and $\widetilde{a}(\cdot,\cdot), \widetilde{\sigma}(\cdot,\cdot)$, and initial values X_0 and $\overline{X}_0$. (Same prob. space, same SBM, same filtration.)
Let $T, \widetilde{T}$ be finite stopping times s.t. $\{X_t : t \in [0,T]\}$ and $\{\overline{X}_t : t \in [0,\widetilde{T}]\}$ are respective strong solutions up to stopping times T, resp. $\widetilde{T}$.
Let $D \subseteq \mathbb{R}^d$ be open, nonempty s.t.

$$\forall t \geq 0 \ \forall x \in D: \quad \widehat{a}(t,x) = a(t,x) \wedge \widetilde{\sigma}(t,x) = \sigma(t,x).$$

Define $\tau := T \wedge \widehat{T} \wedge \inf\{t \geq 0 : X_t \notin D \vee \widehat{X}_t \notin D\}$.
Assume $X_0, \widetilde{X}_0 \in L^2$.

Then $\forall t \geq 0 \ \exists C(t) \in \mathbb{R}_+$ s.t.

$$E\Big(\sup_{s \leq \tau \wedge t} |X_s - \widehat{X}_s|^2\Big) \leq C(t)\, E\big(|X_0 - X_0|^2\big)$$

In particular,

$$P(X_0 = \widetilde{X}_0) = 1 \Rightarrow P(\forall t \leq \tau : X_t = \widehat{X}_t) = 1.$$

Pf: $X_{\tau\wedge t} - \tilde{X}_{\tau\wedge t} = X_0 - \tilde{X}_0 + A_t + M_t$

where
$$A_t := \int_0^t 1_{\tau > s}\left[a(s,X_s) - a(s,\tilde{X}_s)\right] ds$$
$$M_t := \int_0^t 1_{\tau > s}\left[\sigma(s,X_s) - \sigma(s,\tilde{X}_s)\right] dB_s.$$

where containment in D is used to replace $\tilde{a}$ by a, $\tilde{\sigma}$ by σ.

Using argument from last time:

Set $g(t) := E\left(\sup_{s\le\tau\wedge t} |X_s - \tilde{X}_s|^2\right)$

Then
$$g(t) \le 3E(|X_0 - \tilde{X}_0|^2) + 3K^2(t+4)\int_0^t g(s)\,ds.$$

Lemma (Gronwall's lemma) Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be loc. integrable s.t.
$\exists \alpha,\beta \ge 0\ \exists t_0 > 0\ \forall t \in [0,t_0]:\ g(t) \le \alpha + \beta\int_0^t g(s)\,ds$

Then $\forall t \in [0,t_0]:\ g(t) \le \alpha e^{\beta t}$.

So we get $g(t) \le \underbrace{3e^{3K^2(t+4)t}}_{=: C(t).} E(|X_0 - \tilde{X}_0|^2)$

So $P(X = \tilde{X}_0) = 1$ implies $E(\sup_{s \le \tau} |X_t - \tilde{X}_t|^2) = 0.$ $\boxtimes$

Further extensions:

- relax uniform Lipschitz assumption to local Lipschitz assumption.

Corollary: Let $d \in \mathbb{R}$. For each initial value X_0 with $P(X_0 > 0) = 1$, there exists a unique strong solution to

$$dX_t = \frac{d-1}{2X_t} dt + dB_t$$

up to (limit stopping) time $\tau := \lim_{\varepsilon \downarrow 0} \tau_\varepsilon$ where

$$\tau_\varepsilon := \inf\{t \ge 0 : X_t \le \varepsilon\}.$$

When $\tau_\varepsilon < \infty$, the solution extends to X_{τ_1} continuously.

Q: If coefficients are not locally Lipschitz, does strong solution exist? Are they unique?

A: No, not even for ODE's.

To get away from ODE, ask this for

$$dX_t = \sigma(t, X_t)\,dB_t$$

Prop: (Tanaka's SDE). The SDE

$$dX_t = \mathrm{sgn}(X_t)\,dB_t \quad \text{where } \mathrm{sgn}(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

does not admit a strong solution relative to augmented filtration $\widetilde{\mathcal{F}}_t^B := \sigma(\mathcal{N} \cup \mathcal{F}_t)$ where $\mathcal{F}_t := \sigma(B_s : s \leq t)$

Lemma (Tanaka's formula) Let $B = SBM$. Then $\forall t \geq 0$:

$$\boxed{\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{|B_s|<\varepsilon}\, ds \xrightarrow[\varepsilon \downarrow 0]{P} |B_t| - |B_0| - \int_0^t \operatorname{sgn}(B_s)\, dB_s}$$

Pf: Next time:

$$\text{LHS} \text{ "=" } \int_0^t \underbrace{\delta}_{\text{Dirac } \delta}(B_s)\, ds \qquad (L(0))$$