Stochastic differential equations

SDE: $dX_t = U_t\,dt + Y_t\,dB_t$, $U_t = a(t,X_t)$, $Y_t = \sigma(t,X_t)$.

Ex
- $dN_t = \alpha N_t\,dt + \beta N_t\,dB_t$ (population growth)
- $dX_t = \frac{d-1}{2X_t}\,dt + dB_t$ (Bessel process)
- $dX_t = -\nabla V(X_t)\,dt + dB_t$ (Langevin equation)

$\quad$ dB_t: d-dim. SBM

Def A "standard setting" for SDE $dX_t = a(t,X_t)\,dt + \sigma(t,X_t)\cdot dB_t$ is the fourtuple:

$$\left((\Omega,\mathcal{F},P),\ \{\mathcal{F}_t\}_{t\geq 0},\ \{B_t : t \geq 0\},\ X_0 \right)$$

X_0: initial value

where
- P_0-null sets live in $\mathcal{F}_0$
- X_0 is $\mathcal{F}_0$-meas.

and a(.,.), sigma(.,.) are Borel measurable

Def A <u>strong solution</u> to SDE $dX_t = a(t,X_t)\,dt + \sigma(t,X_t)\cdot dB_t$ is stoch process $\{X_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ s.t.:

1) X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and continuous.
2) $\forall t \geq 0$: $\int_0^t |a(s,X_s)|\,ds < \infty \wedge \int_0^t |\sigma(s,X_s)|^2 ds < \infty$ a.s.
3) $\forall t \geq 0$

$$X_t = X_0 + \int_0^t a(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\cdot dB_s \quad \text{a.s.}$$

Note: $|a(t,X_t)| := \|a(t,X_t)\|_2$, $|\sigma(t,X_t)|^2 := \mathrm{Tr}\,\sigma\sigma^T(t,X_t)$

Ex: population growth:

$$d\log N_t = \frac{1}{N_t}\left(\alpha N_t\,dt + \beta N_t\,dB_t\right) - \frac{1}{2}\frac{1}{N_t^2}\beta^2 N_t^2\,dt$$

$$= \left(\alpha - \frac{\beta^2}{2}\right)dt + \beta\,dB_t \Rightarrow N_t = N_0 e^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$$

Ex (Bessel) $d \in \mathbb{N}\setminus\{0\}$: $X_t = |X_0 + B_t|$ (where $B_t \perp\!\!\!\perp X_0$.)

$$dX_t = \frac{d-1}{2X_t}dt + d\tilde{B}_t \qquad \tilde{B}_t = \sum_{i=1}^{d}\int_0^t \frac{1}{X_t}B_s^{(i)}\,dB_s^{(i)} \quad \text{B}$$

NOT strong solution?

Thm (Itô, 1940s) Suppose "standard setting" with $X_0 \in L^2$ and $K \in (0,\infty)$ s.t.

$$\forall t \geq 0 \ \forall x,y \in \mathbb{R}^d: \ |a(t,x)-a(t,y)| + |\sigma(t,x)-\sigma(t,y)| \leq K|x-y|$$

$$\forall t \geq 0: \ |a(t,0)| + |\sigma(t,0)| \leq K$$

Then a strong solution to SDE $dX_t = a(t,X_t)dt + \sigma(t,X_t)dB_t$ exists.

Idea: Picard-Lindelöf iterations:

$$X_t^{(0)} := X_0 \ \wedge \ \forall n \geq 0: \ X_t^{(n+1)} := X_0 + \int_0^t a(s,X_s^{(n)})ds + \int_0^t \sigma(s,X_s^{(n)}) \cdot dB_s$$

with continuous version of integrals on RHS.

First issue: Need to ensure the integrals exist.

Lemma Let $\{Y_t : t \geq 0\}$ be adapted & continuous. Then so are $\{a(t, Y_t) : t \geq 0\}$ and $\{\sigma(t, Y_t) : t \geq 0\}$.

$\boxed{\text{If}}$ $\forall t \geq 0 : \sup_{s \leq t} E|Y_s|^2 < \infty$

then $\forall t \geq 0 : E \int_0^t |a(s, Y_s)|^2 ds < \infty \wedge E \int_0^t |\sigma(s, Y_s)|^2 ds < \infty$

and so $Z_t := \int_0^t a(s, Y_s) ds + \int_0^t \sigma(s, Y_s) \cdot dB_s$

is well defined with a continuous version. s.t. $\forall t \geq 0 : \sup_{s \leq t} E(|Z_s|^2) < \infty$

Pf Assumptions give: $|a(t,x)| + |\sigma(t,x)| \leq K[1 + |x|]$

$$E\left(\sup_{u \leq t} \left| \int_0^u a(s, Y_s) ds \right|^2\right) \leq t E\left(\int_0^t |a(s, Y_s)|^2 ds\right)$$
$$\leq 2t^2K^2 \sup_{s \leq t}(1 + E|Y_s|^2)$$

$$E\left(\sup_{u \leq t} \left| \int_0^u \sigma(s, Y_s) \cdot dB_s \right|^2\right) \overset{\text{Doob-}L^2}{\leq} 4 E\left(\left|\int_0^t \sigma(s, Y_s) \cdot dB_s\right|^2\right)$$
$$\overset{\text{Itô}}{=} 4 E\left(\int_0^t |\sigma(s, Y_s)|^2 ds\right) \leq \ldots \leq 8t^2 \sup_{s \leq t}(1 + E|Y_s|^2)$$

Lemma Define: $g_n(t) := E\left(\sup_{s \le t} |X_s^{(n)} - X_s^{(n-1)}|^2 \right)$ $n \ge 1$

Then $\forall n \ge 0\ \forall t \ge 0$: $g_{n+1}(t) \le 2K^2(t+4) \int_0^t g_n(s)\,ds$

where $g_0(t) := 2K^2 t\,(1 + E(|X_0|^2))$

Pf $X_s^{(n+1)} - X_s^{(n)} = A_t + M_t$ where

$$A_t := \int_0^t [a(s, X_s^{(n)}) - a(s, X_s^{(n-1)})]\,ds$$

$$M_t := \int_0^t [\sigma(s, X_s^{(n)}) - \sigma(s, X_s^{(n-1)})] \cdot dB_s$$

Lipschitz bound gives:

$\underline{n \geq 1}$:

$$E\Big[\sup_{s \le t} A_s^2\Big] \le t\, E\Big(\int_0^t |a(s, X_s^{(n)}) - a(s, X_s^{(n-1)})|^2 ds\Big)$$

$$\le K^2 t\, E\Big(\int_0^t |X_s^{(n)} - X_s^{(n-1)}|^2 ds\Big) \le K^2 t \int_0^t g_n(s)\, ds$$

$$E\Big[\sup_{s \le t} M_s^2\Big] \overset{\text{Doob-}L^2}{\le} 4\, E(M_t^2) \overset{\text{Itô}}{\le} 4\, E\Big(\int_0^t |\sigma(s, X_s^{(n)}) - \sigma(s, X_s^{(n-1)})|^2 ds\Big)$$

$$\le \dots \le 4K^2 \int_0^t g_n(s)\, ds.$$

Now use $E\big(\sup_{s \le t} |X_s^{(n)} - X_s^{(n-1)}|^2\big) \le 2E\big(\sup_{s \le t} |A_s|^2\big) + 2E\big(\sup_{s \le t} |M_s|^2\big)$ $\boxtimes$

$n = 0$ case similar.

$\underline{\text{Pf of Thm}}$ Fix $t > 0$, denote $C(t) = 2K^2(t+4)$.

Iterating gives $g_n(t) \le \dfrac{C(t)^n t^n}{n!}\big[1 + E(|X_0|^2)\big]$

So $P\big(\sup_{s \le t} |X_s^{(n)} - X_s^{(n-1)}| > 2^{-k}\big) \le 4^n g_n(t) \le \dfrac{[4C(t)t]^n}{n!}\big[1 + E(|X_0|^2)\big]$

So $P\Big(\sum_{n \ge 1} \underbrace{\hphantom{xxxxxx}}\ < \infty\Big) = 1$

Define:

$$X_t = \begin{cases} X_0 + \sum_{n=1}^{\infty} \left[X_t^{(n)} - X_t^{(n-1)} \right] & \text{on } \Omega^* \\ X_0 & \text{on } \Omega_*^c \end{cases}$$

Then $X^{(n)} \longrightarrow X_t$ locally uniformly on Ω_*

so X is a continuous process.

We even get:

$$E\left[\sup_{s \le t} |X_s^{(n)} - X_s|^2 \right] \xrightarrow[n\to\infty]{} 0$$

Then Lipschitz cont:

$$\int_0^t a(s, X_s^{(n)})\,ds \xrightarrow[n\to\infty]{L^2} \int_0^t a(s, X_s)\,ds$$

$$\int_0^t \sigma(s, X_s^{(n)})\,dB_s \xrightarrow[n\to\infty]{L^2} \int_0^t \sigma(s, X_s)\,dB_s$$

So, X is strong solution. $\boxed{X}$