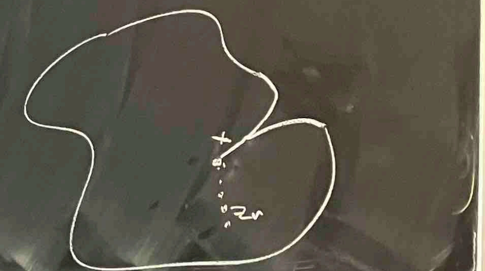
Thm: Let $D \subseteq \mathbb{R}^d$ be open, $\neq \emptyset$. Set $\hat{\tau}_D := \inf\{t>0 : B_t \notin D\}$.

Then $\forall x \in \partial D$: TFAE

1) $P^x(\hat{\tau}_D = 0) = 1$
2) $\forall g : \partial D \to \mathbb{R}$ bdd, meas.: g cont at $x \Rightarrow u_g$ cont. at x.

Here $u_g(z) := E^z\left(g(B_{\tau_D}) \mathbb{1}_{\tau_D < \infty} \right)$



Prop: $\forall x \in \mathbb{R}^d$:

1) $\forall r>0\ \forall \{z_n\} \in D^{\mathbb{N}}$: $z_n \to x$ implies

$$\liminf_{n\to\infty} P^{z_n}\big(\underbrace{\tau_D<\infty \wedge |B_{\tau_D}-x| \le r}_{G_r}\big) \ge P^x(\hat{\tau}_D = 0)$$

2) $\exists \{z_n\} \in D^{\mathbb{N}}$: $z_n \to x$ and

$$\lim_{r\downarrow 0} \limsup_{n\to\infty} P^{z_n}\big(\tau_D<\infty \wedge |B_{\tau_D}-x| \le r\big) = P^x(\hat{\tau}_D = 0)$$

Proof of (1) Denote $E_r := G_r^c$. Then

$$P^z(E_r) \le P^z(\tau_D > \varepsilon) + P^z(\tau_D \le \varepsilon \wedge |B_{\tau_D} - x| > r)$$

Assuming $|z-x| < r/2$,

$$2^{nd}\text{ term} \le P^z\Big(\sup_{t\le\varepsilon} |B_t - z| > r/2\Big) = P^0\Big(\sup_{t\le 1} |B_t| > \varepsilon^{-1/2} r/2\Big) \xrightarrow[\varepsilon\downarrow 0]{} 0 \quad (\forall r>0).$$

For 1^{st}-term, need

$$\tau_{D,\delta} := \inf\{t \ge \delta : B_t \notin D\}. \qquad (\delta > 0).$$

Then $P^z(\tau_D > \varepsilon) \le P^z(\tau_{D,\delta} \ge \varepsilon) \quad (z \in D, \varepsilon > \delta)$

Lemma $\forall t>0\ \forall A \in \sigma(B_s : s \ge t)$: $x \longrightarrow P^x(A)$ is continuous.

Pf Each such A takes the form $\{B_{t+\cdot} \in \tilde{A}\}$ for $\tilde{A} \in \mathcal{B}(\mathbb{R})^{\otimes[0,\infty)}$.

Therefore: $P^x(A) = \int g_t(x-z)\, P^z(B \in \tilde{A})\, dz$

where g_t is density of $\mathcal{N}(0,t)$. BCT $\Rightarrow$ continuity in x. $\boxtimes$

Now: $z_n \to x$ implies

$$P^{z_n}(\tau_{D,\delta} \ge \varepsilon) \xrightarrow[n\to\infty]{} P^x(\tau_{D,\delta} \ge \varepsilon) \xrightarrow[\delta\downarrow 0]{} P^x(\hat{\tau}_D \ge \varepsilon) \xrightarrow[\varepsilon\downarrow 0]{} P^x(\hat{\tau}_D > 0).$$

$\boxtimes$

Proof of 2) Let $F_r := \{\hat{\tau}_D < \infty \wedge |B_{\hat{\tau}_D} - x| \leq r\}$.

Note: $\forall z \in D$: $P^z(F_r) = P^z(G_r)$.

WLOG: • $P^x(\hat{\tau}_D > 0) > 0$. By Blumenthal's 0-1 law: $= 1$.

• $d \geq 2$ because $P^x(\hat{\tau}_D > 0) = 0$ at all $x \in \partial D$.

Now observe:

$$\bigcap_{r>0} F_r = \{\hat{\tau}_D < \infty \wedge B_{\hat{\tau}_D} = x\}$$
$$\supseteq \{\hat{\tau}_D = 0, B_0 = x\}$$

$$\left(\{B_0 = x\} \cap \bigcap_{r>0} F_r\right) \setminus \{\hat{\tau}_D = 0 \wedge B_0 = x\}$$
$$\subseteq \{B_0 = x\} \cap \{\underbrace{\inf\{t > 0 : B_t = x\} < \infty}_{P^x\text{-null} \quad \forall x \in \mathbb{R}^d}\}.$$

Hence:

$$\lim_{r \downarrow 0} P^x(F_r) = P^x\left(\bigcap_{r>0} F_r\right)$$
$$= P^x(\hat{\tau}_D = 0)$$

Since $\hat{\tau}_D$ is stopping time for $\{\mathcal{F}^B_{t+}\}_{t\geq 0}$, SMP gives

$$E^x(\mathbb{1}_{F_r} \mid \mathcal{F}^B_{\hat{\tau}_D\wedge t+}) = P^{B_{\hat{\tau}_D\wedge t}}(F_r)$$

Backward Levy Martingale Convergence Thm:

$$E^x(\mathbb{1}_{F_r} \mid \mathcal{F}^B_{\hat{\tau}_D\wedge t+}) \xrightarrow[t\downarrow 0]{} E^x\Big(\mathbb{1}_{F_r} \Big| \bigcap_{t>0} \mathcal{F}_{\hat{\tau}_D\wedge t+}\Big) \quad P^x\text{-a.s.}$$

$$= E^x(\mathbb{1}_{F_r} \mid \mathcal{F}^B_{0+}) \overset{\text{Blumenthal}}{=} P^x(F_r)$$

Since $\hat{\tau}_D > 0$, P^x-a.s. is assumed, we have

$$\forall r>0: \quad P^{B_t}(F_r) \xrightarrow[t\downarrow 0]{} P^x(F_r) \quad P^x\text{-a.s.} \ (n)$$

On $\{\hat{\tau}_D>0\}$ let B = Brownian path s.t. this holds for all $r\in\{2^{-k}: k\geq 0\}$.

Set $z_n := B_{1/n}$ if $n \geq n_0 := \lceil 1/\hat{\tau}_D \rceil$, and $z_n = B_{1/n_0}$ o/w.

Then

$$\lim_{r\downarrow 0} \limsup_{n\to\infty} P^{z_n}(G_r) = \lim_{k\to\infty} \limsup_{n\to\infty} P^{z_n}(F_{2^{-k}})$$

$$= \lim_{k\to\infty} P^x(F_{2^{-k}}) = P^x(\hat{\tau}_D = 0) \quad \boxtimes$$

Def $x \in \partial D$ is **regular** if $P^x(\hat{\tau}_D = 0) = 1$.

Note in PDE, char. by existence of a **barrier function**.

Corollary: The Dirichlet problem for Laplace equation is solvable in bounded continuous functions on $\bar{D}$ for every bounded boundary condition if and only if every $x \in \partial D$ is regular.

Other PDEs:

Poisson eq: $\Delta u = \rho \qquad u(x) = E^x(u(B_{\tau_D})) + \frac{1}{2} \underbrace{E^x\left(\int_0^{\tau_D} \rho(B_s)\,ds\right)}$

Helmholtz eq. $\frac{1}{2}\Delta u + Ku = \rho \qquad u(x) = E^x\left(u(B_{\tau_D}) e^{K\tau_D}\right) +$

Eigenfunction problem $-\frac{1}{2}\Delta u + V(x)u = \lambda u(x)$

$\forall t \geq 0: \quad u(x) = E^x\left(u(B_t) \exp\left\{\int_0^t (\lambda - V(B_s))\,ds\right\}\right)$

heat equation $\quad \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u \qquad u = u(t,x)$

$$u(t,x) = E^x\left(u(0,B_t)\, 1_{\tau_D > t} \right)$$

Feyman-Kac formula $\quad \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + Vu \qquad V = V(x)$

$$u(t,x) = E^x\left(u(0,B_t) \exp\left\{ \int_0^t V(B_s)\, ds \right\} \right)$$