<u>Thm</u> (SMP for SBM) Let $B = \text{SBM}$, $\{\mathcal{F}_t\} = \text{Br. filtration}$, T = finite stopping time. Then $\{B_{T+t} - B_T : t \geq 0\}$ is SBM that is independent of $\mathcal{F}_{T+}$. Explicitly, writing P^x for the law of SBM with $P^x(B_0 = x) = 1$,

$$\forall x \in \mathbb{R}\ \forall \tilde{A} \in \mathcal{B}(C[0,\infty)) : \ E^x\left(\mathbb{1}_{B_{T+\cdot} \in \tilde{A}} \,\middle|\, \mathcal{F}_{T+}\right) = P^{B_T}(\tilde{A}) \quad P^x\text{-a.s.}$$

<u>Pf</u>: Assume first T is bounded. Set $Z_t := \exp\{i\lambda B_t + \frac{\lambda^2}{2} t\}$.

This is a martingale and since T is bounded

OST: $\forall t \geq u > 0$:

$$E\left(e^{i\lambda B_{T+t} + \frac{\lambda^2}{2}(T+t)} \,\middle|\, \mathcal{F}_{T+u}\right) = e^{i\lambda B_{T+u} + \frac{\lambda^2}{2}(T+u)}$$

Using T is $\mathcal{F}_{T+u}$ meas.:

$$E\left(e^{i\lambda(B_{T+t} - B_{T+u})} \,\middle|\, \mathcal{F}_{T+u}\right) = e^{-\frac{\lambda^2}{2}(t-u)}$$

Iterating: $\forall n \geq 0$: $\forall\ 0 < t_0 < t_1 < \dots < t_n$ $\forall \lambda_1, \dots \lambda_n$:

$$E^x\left(e^{i\sum_{k=1}^{n} \lambda_k (B_{T+t_k} - B_{T+t_{k-1}})} \,\middle|\, \mathcal{F}_{T+t_0} \right) = e^{-\frac{1}{2}\sum_{k=1}^{n} \lambda_k^2 (t_k - t_{k-1})}$$

So $\forall A \in \mathcal{F}_{T+} \subseteq \mathcal{F}_{T+t_0}$,

$$E^x\left(1_A\, e^{i\sum_{k=1}^{n} \lambda_k (B_{T+t_k} - B_{T+t_{k-1}})} \right) = e^{-\frac{1}{2}\sum_{k=1}^{n} \lambda_k^2 (t_k - t_{k-1})} P^x(A) \quad (*)$$

So $\forall A \in \mathcal{F}_{T+}$ s.t. $P^x(A) > 0$:

law of $\{B_{T+t} : t \geq 0\}$ under $P^x(\cdot \mid A)$ is SBM started at B_T

Removing bddness of T: Replace T by $T \wedge n$, $t_0 > 0$ in $(*)$ and take $n \to \infty$ and $t_0 \downarrow 0$, using BCT. $\boxtimes$

Corollary: (Blumenthal's 0-1 law) Let B = SBM and let $\mathcal{F}_t^B := \sigma(B_s : s \leq t)$ and $\mathcal{F}_{t+}^B := \bigcap_{s>t} \mathcal{F}_s^B$. Then

$\forall x \in \mathbb{R}$ $\forall A \in \mathcal{F}_{0+}^B$: $P^x(A) \in \{0, 1\}$.

Pf Let $\tilde{A} \in \sigma(B_s : s \geq 0)$. The SMP:

$$E^x(\mathbb{1}_{\tilde{A}} \mid \mathcal{F}^B_{0+}) = P^x(\tilde{A}) \quad P^x\text{-a.s.}$$

For $A \in \mathcal{F}^B_{0+}$, this means:

$$P^x(A \cap \tilde{A}) = P^x(A) P^x(\tilde{A})$$

Now set $\tilde{A} = A$ to get

$$P^x(A) = P^x(A)^2. \quad \boxtimes$$

<u>Back to Laplace equation:</u>

<u>Thm</u> Let $D \subseteq \mathbb{R}^d$ be non-empty and open. Given $g : \partial D \to \mathbb{R}$ bounded measurable, let $u_g(x) := E^x(g(B_{\tau_D}) \mathbb{1}_{\tau_D < \infty})$.

Let $\hat{\tau}_D := \inf\{t > 0 : B_t \notin D\}$.

Then $\forall x \in \partial D$: TFAE

1) $P^x(\hat{\tau}_D = 0) = 1$

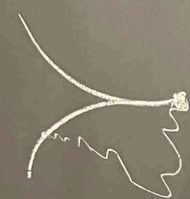
2) $\forall g : \partial D \to \mathbb{R}$ bdd & meas: g cont at $x \Rightarrow u_g$ cont at x

Note: Blumenthal's 0-1 law $\Rightarrow$ $P^x(\underbrace{\hat{\tau}_D = 0}_{\in \mathcal{F}^B_{0+}\text{-meas.}}) \in \{0,1\}$ $\forall x \in \mathbb{R}^d$.

Ex $D = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$

Then $P^0(\hat{\tau}_D = 0) = 0$.

Ex



Key Prop. Let $x \in \partial D$. Then

1) $\forall r > 0$ $\forall \{z_n\} \in D^{\mathbb{N}}$: $z_n \to x$ $\Rightarrow$

$$\liminf_{n \to \infty} P^{z_n}(\tau_D < \infty \wedge |B_{\tau_D} - x| \le r) \ge P^x(\hat{\tau}_D = 0)$$

2) $\exists \{z_n\} \in D^{\mathbb{N}}$: $z_n \to \infty$ s.t.

$$\lim_{r \downarrow 0} \limsup_{n \to \infty} P^{z_n}(\tau_D < \infty \wedge |B_{\tau_D} - x| \le r) = P^x(\hat{\tau}_D = 0)$$

Pf of Thm from Prop: (1) ⇒ (2)

Let $E_r = \{\tau_D = \infty\} \cup \{\tau_D < \infty \wedge |B_{\tau_D} - x| > r\}$.

Prop: $z_n \to x$, $P^{z_n}(E_r) \to 0$ $\forall r > 0$.

$$|u_g(z_n) - g(x)| \leq E^x\left(|g(B_{\tau_D}) 1_{\tau_D < \infty} - g(x)|\right)$$

$$\leq \sup_{z \in B(x,r) \cap \partial D} |g(z) - g(x)| + 2\|g\|_\infty P^{z_n}(E_r)$$

Taking $n \to \infty$ and $r \downarrow 0$, the RHS $\to 0$ (by cont of g at x).

$\boxed{\neg(1) \Rightarrow \neg(2)}$ Assume $P^x(\hat{\tau}_D > 0) > 0$. Let

$$g_r(z) = \max\left\{1, \frac{|z-x|}{r}\right\}$$

Let $h_r(z) = u_{g_r}(x) + P^z(\tau_D = \infty)$. By ≥ 0 of all terms:

$$h_r(z) \geq E^z\left(g_r(B_{\tau_D}) 1_{\tau_D < \infty \wedge |B_{\tau_D} - x| > r}\right) + P^z(\tau_D = \infty)$$

$$= P^z(E_r)$$

By Prop $\exists z_n \to x$ s.t. $\lim_{r \downarrow 0} \liminf_{n \to \infty} P^{z_n}(E_r) = P^x(\hat{\tau}_D > 0) > 0$

So $\exists r>0$ $\liminf_{n\to\infty} h_r(z_n) > 0$ yet $h_r(x) = 0$,

meaning h_r is NOT continuous at x !!

Then either U_{g_r} is NOT continuous or U_1 is NOT continuous at x

$\boxtimes$