Dirichlet problem for Laplace eq.

Notation $\Delta f(x) = \sum_{k=1}^{d} \frac{\partial^2 f}{\partial x_k^2}$ $f \in C^2(\mathbb{R}^d)$

Given $D \subseteq \mathbb{R}^d$ nonempty open,

Def: $u : D \to \mathbb{R}$ solves Dirichlet problem for Laplace equation with boundary values $g : \partial D \to \mathbb{R}$ if

1) $u \in C^2(D) \wedge \forall x \in D : \Delta u(x) = 0$

2) $\forall x \in \partial D : \lim_{\substack{z \to x \\ z \in D}} u(z) = g(x)$

1) is equivalent to MVP (harmonic function)

$\forall x \in D \; \forall r > 0 : \overline{B(x,r)} \subseteq D \Rightarrow u(x) = \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} u(z) \, dz$

2) $g \in C(\partial D)$ ... we may assume $u \in C^2(D) \cap C(\overline{D})$.

Key point: Solution can be written using exit distribution of SBM

Thm (Kakutani 1944) Let $D \subseteq \mathbb{R}^d$ be open, nonempty. For $\{B_t : t \geq 0\}$ SBM with $P^x(B_0 = x) = 1$ define

$$\tau_D := \inf\{t \geq 0 : B_t \notin D\}.$$

Then τ_D is a stopping time with $B_{\tau_D} \in \partial D$ on $\{\tau_D < \infty\}$.

If $u \in C^2(D) \cap C(\bar{D})$ obeys $\Delta u(x) = 0$ and is bounded, then

$$\forall x \in D : P^x(\tau_D < \infty) = 1$$

implies

$$\forall x \in D : u(x) = E^x\big(u(B_{\tau_D})\big)$$

Pf $D_n := \{x \in D : \mathrm{dist}(x, D^c) > 1/n,\ \mathrm{dist}(0, x) < n\}$.

Then u has bounded derivatives on D_n so Itô formula.

$$du(B_{t\wedge\tau_{D_n}}) = 1_{\tau_{D_n} > t} \nabla u(B_{t\wedge\tau_{D_n}}) \cdot dB_t \qquad \left(\sum_{i,j=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j}\, d\langle B^{(i)}, B^{(j)}\rangle_t\right)$$

$$+ \frac{1}{2} 1_{\tau_D > t} \underbrace{\Delta u}_{=0}(B_{t\wedge\tau_{D_n}})\, dt$$

So $M_t := u(B_{t\wedge\tau_{D_n}})$ is loc. martingale.

Since M is bounded, it is a martingale.

(By (*)) we may use OST to conclude

$$u(x) = E^x(M_0) = E^x(M_{\tau_{D_n}}) = E^x\left(u(B_{\tau_{D_n}}) 1_{\tau_{D_n} < \infty}\right)$$

As $n \to \infty$, $(*)$ implies $\tau_{D_n} \to \tau_D < \infty$ a.s.

This implies $B_{\tau_{D_n}} \to B_{\tau_D}$ a.s.

Since u is bounded, BCT implies $E^x(u(B_{\tau_{D_n}})) \to E^x(u(B_{\tau_D}))$

+ continuity of u on $\overline{D}$. $\boxtimes$

<u>Note</u> Boundedness of u essential: $D = \{x \in \mathbb{R}^2 : |x| \geq 1\}$

$u(x) = \log|x|$ is harmonic on

Recurrence to balls of $d=2$ SBM $\Rightarrow \tau_D < \infty$ a.s.

Yet $u(x) = 0 \quad \forall x \in \partial D$

So can't satisfy the conclusion of Thm!

Corollary: If $u \in C^2(D) \cap C(\bar{D})$ is <u>bounded</u> and harmonic in D,

then $$\sup_{x \in D} |u(x)| \le \sup_{x \in \partial D} |u(x)|$$ (Maximum principle)

wherever (*) holds.

Corollary Under (*), the Dirichlet problem for Laplace eq. has at most one bounded solution.

Poisson kernel: $E^x(g(B_{\tau_D})) = \int g(z)\, H^D(x, dz)$ ← harmonic measure

$$H^D(x, A) := P^x(B_{\tau_D} \in A)$$

Q: What if (*) fails?

Prop Suppose $D \subseteq \mathbb{R}^d$ is open, nonempty and $g : \partial D \to \mathbb{R}$ measurable. Then $\forall a \in \mathbb{R}$:

$$u(x) := E^x\left(g(B_{\tau_D}) 1_{\tau_D < \infty}\right) + a P^x(\tau_D = \infty)$$

is harmonic on D with $u(x) = g(x)$ on ∂D.

Thm (Strong Markov property) Let T be a finite stopping time for SBM with filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Set
$$\mathcal{F}_{T+} := \{ A \in \mathcal{F} : (\forall t \geq 0 : A \cap \{T \leq t\} \in \mathcal{F}_{t+}) \}$$
where $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$. Then
$$\{ B_{T+t} - B_T : t \geq 0 \}$$
is SBM independent of $\mathcal{F}_{T+}$.

Proof of Prop: Let $u(x) := E^x(g(B_{\tau_D}) 1_{\tau_D < \infty})$.

Let $r > 0$ be s.t. $\overline{B(x,r)} \subseteq D$. Set $T = \tau_{B(x,r)}$.

Denote $W_t := B_{T+t} - B_T$. Then SMP $\Rightarrow$ W is SBM $\perp\!\!\!\perp \mathcal{F}_T$.

Define $\tau_D^W := \inf\{t \geq 0 : W_t + B_T \notin D\}$. Then
$$\tau_D^W + T = \tau_D \quad \wedge \quad B_{\tau_D} = B_T + W_{\tau_D^W} \text{ on } \tau_D < \infty.$$

Hence $u(x) = E^x\left(g(B_{\tau_D}) 1_{\tau_D < \infty}\right)$

$$= E^x\left(g\left(B_T + W_{\tau_D^W}\right) 1_{\tau_D^W < \infty}\right)$$

$$= E^x\left(\underbrace{E^x\left(g\left(\cdot + W_{\tau_D^W}\right) 1_{\tau_D^W < \infty} \mid \mathcal{F}_T\right)\Big|_{\cdot = B_T}}_{u(B_T)}\right)$$

So we get

$$u(x) = E^x\left(u\left(B_{\tau_{B(x,r)}}\right)\right)$$

$$= \int u(x+ru)\, \mu(du)$$ where μ = uniform distribution on $\partial B(0,1)$.

So u obeys MVP and so is harmonic.

For the second term, use

$$a P^x(\tau_D = \infty) = a - E^x\left(a \cdot 1_{\tau_D < \infty}\right) \qquad \boxtimes$$

So uniqueness may fail when $(*)$ ~~holds~~ fails.