Recurrence & transience of Bessel processes

Bessel process: $X_t = X_0 + \int_0^t \frac{d-1}{2X_s} 1_{\tau_0 > s} \, ds + B_{t \wedge \tau_0}$

where $\tau_0 := \inf\{t \geq 0 : X_t = 0\}$. $(X_0 > 0)$

SDE: $dX_t = 1_{\tau_0 > t}\left(\frac{d-1}{2X_t} dt + dB_t\right)$

Today: Study τ_0 for X and d-dim SBM.

P^x = law of $\{X_t : t \geq 0\}$ with $P^x(X_0 = x) = 1$.

Last time: $\phi_d(x) = \begin{cases} x^{2-d} & d \neq 2 \\ \log x & d = 2 \end{cases}$ $(X_0 > 0)$

Then $\{\phi(X_t) : t \geq 0\}$ loc. martingale.

Lemma Let $0 < a < x < b$. Set $\tau_a := \inf\{t \geq 0 : X_t = a\}$. Then

$$P^x(\tau_a \wedge \tau_b < \infty) = 1$$

$$P^x(\tau_a < \tau_b) = \frac{\phi_d(b) - \phi_d(x)}{\phi_d(b) - \phi_d(a)}.$$

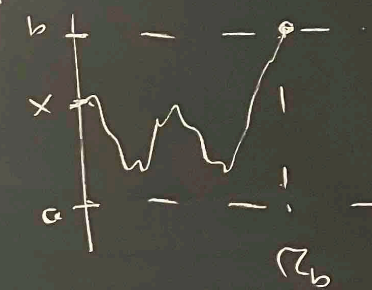
Intermezzo on SBM

Lemma Let P^x = laws of SBM (on $\mathbb{R}$) with $P^x(B_0 = x) = 1$.

Then $\forall \; a < x < b$:

$$P^x(\tau_a \wedge \tau_b < \infty) = 1$$

$$P^x(\tau_a < \tau_b) = \frac{b-x}{b-a}.$$



Pf: $P(\tau_a \wedge \tau_b = \infty) \leq P\left(\bigcap_{n\geq 0} \{|B_{n+1} - B_n| \leq b-a\}\right)$

$$= \prod_{n \geq 0} P(|N(0,1)| \leq b-a) = 0$$

Next: $\{B_t : t \geq 0\}$ is martingale

$$E(B_t | \mathcal{F}_s) \underset{s<t}{=} E(B_s + B_t - B_s | \mathcal{F}_s) = B_s + 0 = B_s.$$

OST $M_t = B_{t \wedge \tau_a \wedge \tau_b}$ is bounded martingale.

$$x = E^x(M_0) \overset{OST}{=} E^x(M_{\tau_a \wedge \tau_b})$$

$$= P^x(\tau_a < \tau_b)\, a + P^x(\underset{1 - P^x(\tau_a < \tau_b)}{\underbrace{\tau_a > \tau_b}})\, b$$

$\boxtimes$

Pf Let $\tilde{B}$ be SBM independent of B and X.

Define $$Z_t = \phi_d(X_0) + \int_0^t 1_{\tau_a \wedge \tau_b > s} \phi_d'(X_s)\, dB_s + 1_{t \geq \tau_a \wedge \tau_b} (\tilde{B}_t - \tilde{B}_{\tau_a \wedge \tau_b})$$

Note $\forall t \leq \tau_a \wedge \tau_b$: $Z_t = \phi_d(X_t)$ a.s.

$$\forall t \geq 0: \langle Z \rangle_t = \int_0^t \left[1_{\tau_a \wedge \tau_b > s} \phi'(X_s)^2 + 1_{\tau_a \wedge \tau_b \leq s} \right] ds$$

Defining: $T(t) := \inf\{u \geq 0 : \langle Z \rangle_u \geq t\}$

we have $T(t) < \infty$ a.s. $\wedge$ $T(t) \to \infty$ as $t \to \infty$.

Time change argument: $\{ Z_{T(t)} : t \geq 0 \}$ is SBM (modulo null set)

Note also: $\exists c_1, c_2 \in (0, \infty) \mid \forall t \geq 0$: $c_1 t \leq T(t) \leq c_2 t$

Then $$P(\tau_a \wedge \tau_b = \infty) = P\left(\forall t \geq 0: Z_{T(t)} \in \left(\phi_d(a) \wedge \phi_d(b), \phi_d(a) \vee \phi_d(b) \right) \right)$$
$$= 0$$

Proceeding as before. $M_t := \phi_d(X_{t\wedge\tau_a\wedge\tau_b})$ is bdd martingale.

OST: $\phi_d(x) = E^x(M_0) = E^x(M_{\tau_a\wedge\tau_b})$

$= \phi_d(a)\,P^x(\tau_a<\tau_b) + \phi_d(b)\,P^x(\tau_a>\tau_b)$

Use $P^x(\tau_a<\tau_b)+P^x(\tau_a>\tau_b)=1$ to get the formula. $\boxtimes$

<u>Thm</u> Let X = d-dim Bessel process. with $P^x(X_0=x)=1$, $x>0$. Then

1) $\boxed{d>2}$ $\tau_0=+\infty$ a.s, $\inf_{t\geq 0} X_t>0$ a.s, $\lim_{t\to\infty} X_t=+\infty$ a.s.

2) $\boxed{d<2}$ $\tau_0<+\infty$ a.s, $\sup_{t\geq 0} X_t<\infty$ a.s, $\lim_{t\to\infty} X_t=0$ a.s.

3) $\boxed{d=2}$ $\tau_0=+\infty$ a.s, $\inf_{t\geq 0} X_t=0$ a.s, $\sup_{t\geq 0} X_t=\infty$ a.s.

$\liminf_{t\to\infty} X_t=0$ a.s. $\limsup_{t\to\infty} X_t=+\infty$ a.s.

Pf As X is continuous, we have

$$\tau_b \xrightarrow[b \to \infty]{} +\infty \quad a.s.$$

$\boxed{d>2}$ $\phi_d(x) = x^{2-d}$ $\to 0$ $x \to +\infty$; $\to +\infty$ $x \downarrow 0$.

$$P^x(\tau_a < \infty) = \lim_{b\to\infty} P^x(\tau_a < \tau_b)$$

$$= \lim_{b\to\infty} \frac{\phi_d(b) - \phi_d(x)}{\phi_d(b) - \phi_d(a)} = \frac{\phi_d(x)}{\phi_d(a)} \xrightarrow[a\downarrow 0]{} 0$$

Let $\{a_n\}$ be s.t.

$$P^x(\tau_{a_n} < \infty) \le 2^{-n}. \quad BC. \quad P^x(\tau_{a_n} < \infty \text{ i.o.}) = 0.$$

and hence $P^x(\inf_{t\ge 0} X_t > 0) = 1$.

$\boxed{d=2}$ $$P^x(\tau_a < \tau_b) = \frac{\log b - \log x}{\log b - \log a}$$

$$P(\inf_{t\ge 0} X_t = 0) = 1$$

$$P^x(\tau_0 < \infty) = 0$$

So $$P^x(\tau_a < \infty) = \lim_{b\to\infty} P^x(\tau_a < \tau_b) = 1$$

$$P^x(\tau_0 < \infty) = \lim_{b\to\infty} P^x(\tau_0 < \tau_b) \le \lim_{b\to\infty} \underbrace{\lim_{a\downarrow 0} P^x(\tau_a < \tau_b)}_{=0}$$

τ_a

Corollary Let $d \in \mathbb{Z}$, $d \geq 1$, and B = d-dimensional SBM.

(1) $(d=1)$ B is recurrent to points: $\{B_t : t \geq 0\} = \mathbb{R}$ a.s.

(2) $(d=2)$ B is NOT —— " ——: $\forall x \neq 0$: $P(\exists t \geq 0 : B_t = x) = 0$.

yet it is recurrent to open balls: $\{B_t : t \geq 0\}$ is dense in $\mathbb{R}^2$ a.s.

(3) $(d \geq 3)$ B is NOT recurrent to open balls: $\forall x \in \mathbb{R}^d \setminus \{0\}$: $\inf_{t \geq 0} |B_t - x| > 0$ a.s. (x)

Pf: $R_t := |B_t - x|$ is d-dim Bessel process started at $|x|$. $\boxtimes$