Stochastic processes

$\{X_t : t \in T\}$ $\quad T$ = index set, $\quad X_t \in \mathcal{X}$ ← set of values

Q: What describes this family uniquely?

Def Given $\{X_t : t \in T\}$ with values in $(\mathcal{X}, \Sigma)$, for each $n \geq 1$ and each $t_1 \dots t_n \in T$, define

$$\mu_{(t_1 \dots t_n)}(A) := P((X_{t_1} \dots X_{t_n}) \in A) \quad A \in \Sigma^{\otimes n}$$

The family $\{\mu_{\underline{t}} : \underline{t} \in T^n, n \geq 1\}$ are **finite-dimensional distributions** of $\{X_t : t \in T\}$ (FDD)

Q: When do FDD correspond to a stoch. process?

Def A family of prob measures $\{\mu_{\underline{t}} : \underline{t} \in T^n, n \geq 1\}$ with $\mu_{(t_1 \dots t_n)}$ being a measure on $(\mathcal{X}^n, \Sigma^{\otimes})$ is **consistent** if

1) $\forall n \geq 1 \ \forall \sigma \in S_n \ \forall t_1 \dots t_n \ \forall A_1 \dots A_n$:

$$\mu_{(t_{\sigma(1)} \dots t_{\sigma(n)})}(A_{\sigma(1)} \times \dots \times A_{\sigma(n)}) = \mu_{(t_1 \dots t_n)}(A_1 \times \dots \times A_n)$$

2) $\forall n \geq 1 \ \forall A_1 \dots A_{n-1} \in \Sigma \ \forall t_1 \dots t_n \in T$:

$$\mu_{(t_1 \dots t_n)}(A_1 \times \dots \times A_{n-1} \times \mathcal{X}) = \mu_{(t_1 \dots t_{n-1})}(A_1 \times \dots \times A_{n-1})$$

Ex For Poisson process, consistency follows from:

$$\text{Poisson}(\lambda) \oplus \text{Poisson}(\mu) = \text{Poisson}(\lambda+\mu)$$

For Brownian motion:

$$\mathcal{N}(0,s) \oplus \mathcal{N}(0,t) = \mathcal{N}(0,t+s)$$

Q: Does every consistent family of FDD correspond to process?

A: We need $\mathcal{X}$ to be "nice".

Def ("Nice") A measurable space $(\mathcal{X}, \Sigma)$ is standard Borel if $\mathcal{X}$ is completely-metrizable second countable topological space and Σ = assoc. Borel sets.

Ex $\mathbb{R}, \mathbb{R}^n, \ell^p(\mathbb{N})$ $(p \in [1,\infty))$

$C([0,1],\mathbb{R})$, $C([0,\infty),\mathbb{R})$

$L^p(\mathbb{R})$ $p \in [1,\infty)$.

Thm (Kolmogorov Extension Thm)

Let $(\mathcal{X}, \Sigma)$ be standard Borel and $T \neq \emptyset$ a set.
Given any consistent family $\{\mu_{\underline{t}} : \underline{t} \in T^n, n \geq 1\}$ of prob. measures ($\mu_{(t_1 \ldots t_n)}$ is a measure on $(\mathcal{X}^n, \Sigma^{\otimes n})$)
there exists a prob. space $(\Omega, \mathcal{F}, P)$ and a stoch. process $\{X_t : t \in T\}$ on it s.t.

$$\forall n \geq 1 \; \forall t_1 \ldots t_n \in T \; \forall A \in \Sigma^{\otimes n} : P((X_{t_1} \ldots X_{t_n}) \in A) = \mu_{t_1 \ldots t_n}(A)$$

<u>Key idea I</u>: Measure extension solved in very abstract way.

Thm (Hahn-Kolmogorov) Let $\mathcal{A}$ = algebra of subsets of Ω and $\mu : \mathcal{A} \mapsto [0, \infty]$ s.t. $\mu(\emptyset) = 0$ and

1) finite additivity: $\forall A, B \in \mathcal{A} : A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$
2) countably subadditivity: $\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$: disjoint $\wedge \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ $\Rightarrow \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$

Then $\exists \bar{\mu}$ = measure on $(\Omega, \sigma(\mathcal{A}))$ s.t.

$$\forall A \in \mathcal{A} : \bar{\mu}(A) = \mu(A)$$

STEP 1 $\Omega := \mathcal{X}^T = \{\text{functions } T \to \mathcal{X}\}$

$$\mathcal{F} := \Sigma^{\otimes T} := \sigma\left(\left\{ \bigtimes_{t\in T} A_t : (\forall t \in T : A_t \in \Sigma) \wedge \underbrace{\{t \in T : A_t \neq \mathcal{X}\}}_{\subseteq S} \text{ finite} \right\}\right)$$

Need: $\mathcal{F}_S :=$ - - - - - - - - - - - - -

Let $\mathcal{A} = \bigcup_{\substack{S \subseteq T \\ \text{finite}}} \mathcal{F}_S$

Lemma $\mathcal{A}$ is an algebra on Ω.

For $S \subseteq T$ finite set $S = \{t_1 \ldots t_n\}$.

$$\forall A \in \mathcal{F}_S : \quad P(A) = \mu_{(t_1 \ldots t_n)}(\pi_S(A))$$

where $\pi_S : \Omega \to \mathcal{X}^S$ is coordinate proj.

Lemma P is well defined and finitely additive

STEP 2 Topology enters.

Prop $\forall \{B_n\}_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$: $B_n \downarrow \emptyset \Rightarrow P(B_n) \downarrow 0$.

Key lemma Let $(\mathcal{X}, \Sigma)$ = standard Borel, μ = finite measure.

Thm $\forall B \in \Sigma$: $\mu(B) = \sup\{\mu(C) : C \subseteq B,\ \text{compact}\}$

(inner regularity)