last time: Levy char. of SBM:

$$X_t = \sum_{k=1}^{n} \int_0^t Y_s^{(k)} dB_s^{(k)}$$ continuous

continuous

loc. martingale enough.

$\Rightarrow$ $\forall t \geq 0: \langle X \rangle_t = t \Rightarrow X$ is SBM.

Ex $R_t := \left(\sum_{k=1}^{d} (B_t^{(k)})^2 \right)^{1/2}$ where $B^{(1)}, \dots B^{(d)}$ are iid SBM.

Then $dR_t = \frac{d-1}{2R_t} dt + \underbrace{\frac{1}{R_t} \sum_{k=1}^{d} B_t^{(k)} dB_t^{(k)}}_{d\tilde{B}_t}$

Lemma $\tilde{B}_t := \sum_{k=1}^{d} \int_0^t \frac{1}{R_s} B_s^{(k)} dB_s^{(k)}$ is SBM.

Pf Cont paths (by choice)

$$\langle \tilde{B} \rangle_t = \sum_{k=1}^{d} \int_0^t \frac{1}{R_s^2} (B_s^{(k)})^2 ds = \int_0^t 1 \, ds = t.$$

Lévy char. $\Rightarrow$ $\tilde{B}$ is SBM $\boxtimes$

(Bessel's SDE)

So we showed that R_t obeys the SDE: $dR_t = \frac{d-1}{2R_t} dt + d\tilde{B}_t$.

Q: What if $\langle X \rangle$ is general?

A: Reparametrize time so that $\langle X \rangle$ is linear and get same conclusion.

Thm (Time change to SBM) Let X be continuous s.t. $\exists\, Y \in \mathcal{V}_{loc}$

$$\forall t \geq 0: X_t = \int_0^t Y_s \, dB_s$$

Assume in addition that $\int_0^\infty Y_s^2 \, ds = +\infty$ a.s. Set

$$\forall t \geq 0: \quad T(t) := \inf\left\{ u \geq 0 : \int_0^u Y_s^2 \, ds \geq t \right\}.$$

Then $T(t)$ is a stopping time with $T(t) < \infty$ a.s. $(\forall t \geq 0)$ and $T(t) \xrightarrow[t \to \infty]{} \infty$ a.s. and $\{X_{T(t)} : t \geq 0\}$ is SBM, modulo change on a null set.

Intermezzo on stopped martingales

Def Let T be a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F})$.

$$\mathcal{F}_T := \left\{ A \in \mathcal{F} : \left(\forall t \geq 0 : A \cap \{T \leq t\} \in \mathcal{F}_t \right) \right\}$$

are events measurable by stopping time T.

Note. We do allow $T = +\infty$.

- $\mathcal{F}_T$ is NOT $\mathcal{F}_t$ with $t := T$!

Lemma Let T, S be stopping times for $\{\mathcal{F}_t\}_{t\geq 0}$. Then

1) $\mathcal{F}_T$ (and $\mathcal{F}_S$) is σ-alg, and T is $\mathcal{F}_T$-measurable

2) $\mathcal{F}_T \cap \mathcal{F}_S = \mathcal{F}_{T\wedge S}$ and so $S \leq T \Rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$.

Pf: (1) Easily, $\mathcal{F}_T$ closed under countable unions.

Complements: $A \in \mathcal{F}_T$ then

$$(A^c \cap \{T\leq t\})^c = A \cup \{T>t\} = (A \cap \underbrace{\{T\leq t\}}_{\in \mathcal{F}_t}) \cup \underbrace{\{T>t\}}_{\in \mathcal{F}_t}$$

$$\Rightarrow A^c \cap \{T\leq t\} \in \mathcal{F}_t.$$

$$\{T\leq u\} \cap \{T\leq t\} = \{T \leq t\wedge u\} \in \mathcal{F}_{t\wedge u} \in \mathcal{F}_t$$

$\Rightarrow \forall u \in \mathbb{R}: \{T\leq u\} \in \mathcal{F}_T$. So $\{T\in A\} \in \mathcal{F}_T \ \forall A \in \mathcal{B}(\mathbb{R})$. $\boxtimes$

(2) HW exercise. $\boxtimes$

Q: If $\{X_t : t\geq 0\}$ is adapted to $\{\mathcal{F}_t\}$ is X_T $\mathcal{F}_T$-meas?

A: Needs assumptions on X.

General sufficient condition:

Def: An $\mathbb{R}$-val'd stoch process $\{X_t : t \geq 0\}$ is <u>progressively measurable</u> (w.r.t. $\{\mathcal{F}_t\}$) if

$$\forall t \geq 0 \ \forall A \in \mathcal{B}(\mathbb{R}) : \ \{(s,\omega) \in [0,t] \times \Omega : X_s(\omega) \in A\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$$

Suffices to have one sided continuity:

Lemma Suppose X is adapted and left-continuous. Then for any finite stopping time, X_T is $\mathcal{F}_T$-measurable

<u>Pf</u> HW.

<u>Note</u> $(X_T)(\omega) := X_{T(\omega)}(\omega)$

<u>If</u> M is a martingale, then $\forall s \le t$: $E(M_t | \mathcal{F}_s) = M_s$.

<u>Q</u>: Does this work for stopping times?

<u>Thm</u> (OST) Let $M = \{M_t : t \geq 0\}$ be continuous martingale and $S \leq T$ stopping times. If

1) either T is bounded $(T \in L^\infty)$
2) or $T < \infty$ a.s. and $\{M_t : t \geq 0\}$ is UI

then $M_T \in L^1$ and

$$E(M_T | \mathcal{F}_S) = M_S \quad \text{a.s.}$$

<u>Pf</u> Strategy: truncate & discretize (divide & conquer)

Assume $\exists \underset{\text{constant}}{K} \in (0, \infty)$: $S \leq T \leq K$.

In addition, assume $\exists \Sigma \subseteq [0, K]$ finite s.t. $T(\omega), S(\omega) \in \Sigma$.

Then $\forall t \in \Sigma$: $\{T = t\} = \{T \leq t\} \setminus \bigcup_{t' \in \Sigma} \{T \leq t'\} \in \mathcal{F}_t$

So $|M_T| \leq \sum_{t \in \Sigma} |M_t| \in L^1$

and $\forall A \in \mathcal{F}_T$:

$$E(M_T 1_A) = \sum_{t \in \Sigma} E(M_t 1_{\underbrace{A \cap \{T=t\}}_{\in \mathcal{F}_t}}) \qquad M_t = E(M_T | \mathcal{F}_t)$$

$$= \sum_{t \in \Sigma} E(M_K 1_{A \cap T=t}) = E(M_K 1_A)$$

So $M_T = E(M_K | \mathcal{F}_T)$ and similarly $M_S = E(M_K | \mathcal{F}_S)$.

"Smaller always wins": $E(M_T | \mathcal{F}_S) = E(E(M_K | \mathcal{F}_T) | \mathcal{F}_S) = E(M_K | \mathcal{F}_S)$

$$= M_S.$$

Now take T, S general (still bounded by K).

Discretize: $T_N := 2^{-N} \lceil 2^N T \rceil$, $S_N := 2^{-N} \lceil 2^N S \rceil$.

Then $S_N \leq T_N \leq K+1$ $(N \geq 0)$ So

$$M_{S_N} = E(M_{K+1} | \mathcal{F}_{S_N}), \quad M_{T_N} = E(M_{K+1} | \mathcal{F}_{T_N})$$

So $\{M_{T_N} : N \geq 1\}$ is UI. Now for $A \in \mathcal{F}_S \subseteq \mathcal{F}_{S_N} \subseteq \mathcal{F}_{T_N}$:

$$E(M_{T_N} 1_A) = E(M_{K+1} 1_A) = E(M_{S_N} 1_A) + L^1 \text{ conv.}$$

gives $E(M_T 1_A) = E(M_S 1_A) \Rightarrow M_T \in L^1 \wedge E(M_T | \mathcal{F}_S) = M_S$. $\boxtimes$