Itô formulas & Lévy characterization

$f \in C^2(\mathbb{R})$

last time X - semimartingale, $df(X_t) = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,d\langle X\rangle_t$

extends to other types of functions:

$f(X_t, \langle X\rangle_t)$

Then for f type C^2 and C^1:

$$df(X_t, \langle X\rangle_t) = \frac{\partial f}{\partial x_1}(X_t, \langle X\rangle_t)\,dX_t + \left(\frac{\partial f}{\partial x_2}(X_t, \langle X\rangle_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(X_t, \langle X\rangle_t)\right) d\langle X\rangle_t$$

Ex $$Z_t = \exp\left\{ \underbrace{\int_0^t Y_s\,dB_s}_{X_t} - \frac{1}{2} \underbrace{\int_0^t Y_s^2\,ds}_{\langle X\rangle_t} \right\} \qquad Y \in V^{loc}$$

$$dZ_t = Z_t Y_t\,dB_t + \left(-\frac{1}{2} Y_t^2\,dt + \frac{1}{2} Y_t^2\,dt\right)$$

$$= Z_t Y_t\,dB_t$$

$$Z_t = 1 + \int_0^t Z_s Y_s\,dB_s$$

so $\{Z_t : t \geq 0\}$ is a local martingale

generalizes $M_t = e^{B_t - \frac{t}{2}}$

Rules $\quad dX_t\,dt = 0, \quad dX_t\, d\langle X\rangle_t = 0, \quad (dX_t)^2 = d\langle X\rangle_t.$

Thm Let $B^{(1)}, \dots, B^{(d)}$ be indep. SBMs with common Brownian filtration. Given $\{Y_t^{i,k} : t \geq 0\}_{\substack{i=1\dots m\\ k=1\dots d}} \in \mathcal{V}^{loc}$ set

$$X_t^{(i)} := X_0^{(i)} + \sum_{k=1}^{d} \int_0^t Y_s^{(k,i)}\, dB_s^{(k)} \qquad i = 1 \dots m.$$

Let $f \in C(\mathbb{R}^m)$. Then $f \circ \underbrace{(X^{(1)} \dots X^{(m)})}_{X}$ is a semimartingale with

$$df(X_t) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(X_t) \sum_{k=1}^{d} Y_t^{k,i}\, dB_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_t) \sum_{k=1}^{d} Y_t^{(k,i)} Y_t^{(k,j)}\, dt$$

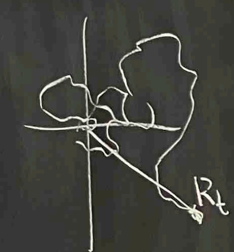
Rules $\quad dB_t^{(k)}\, dB_t^{(\ell)} = \delta_{k,\ell}\, dt$

Ex Let $B^{(1)} \dots B^{(d)}$ be iid SBM, let $R_t := \left(\sum_{k=1}^{d} [B_t^{(k)}]^2\right)^{1/2}$.

Denote $f(x_1 \dots x_d) = \left(\sum_{k=1}^{d} x_k^2\right)^{1/2}$

$$\frac{\partial f}{\partial x_k} = \frac{x_k}{f(x)}, \qquad \frac{\partial^2 f}{\partial x_k^2} = \frac{1}{f(x)} - \frac{x_k^2}{f(x)^3}$$

$$dR_t = df(B_t^{(1)} \dots B_t^{(d)}) = \sum_{k=1}^{d} \frac{B_t^{(k)}}{R_t}\, dB_t^{(k)} + \frac{1}{2} \sum_{k=1}^{d} \left(\frac{1}{R_t} - \frac{(B_t^{(k)})^2}{R_t^3} \right) dt$$



$$dR_t = \frac{1}{R_t} \sum_{k=1}^{d} B_t^{(k)}\, dB_t^{(k)} + \frac{d-1}{2} \frac{1}{R_t}\, dt$$

Product rule

Thm Let $X, \tilde{X}$ be semimartingales. Then so is $\{ X_t \tilde{X}_t : t \geq 0 \}$ and

$$d(X_t \tilde{X}_t) = \tilde{X}_t \, dX_t + X_t \, d\tilde{X}_t + d\langle X, \tilde{X} \rangle_t$$

where

$$\langle X, \tilde{X} \rangle_t := \frac{1}{4} \langle X + \tilde{X} \rangle_t - \frac{1}{4} \langle X - \tilde{X} \rangle_t$$

cross variation process

$$\boxed{\text{Rule } \; dX_t \, d\tilde{X}_t = d\langle X, \tilde{X} \rangle_t}$$

Pf $d(X_t \pm \tilde{X}_t)^2 \overset{\text{Itô}}{=} 2(X_t \pm \tilde{X}_t)(dX_t \pm d\tilde{X}_t) + d\langle X \pm \tilde{X} \rangle_t$

Now use $X_t \tilde{X}_t = \frac{1}{4}(X_t + \tilde{X}_t)^2 - \frac{1}{4}(X_t - \tilde{X}_t)^2$ $\boxtimes$

Lemma: Let $\Pi_n = \{ 0 = t_0^n < \ldots < t_{m(n)}^n = t \}$. Then

$$\sum_{i=1}^{m(n)} (X_{t_i} - X_{t_{i-1}})(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}) \xrightarrow[\|\Pi_n\| \to 0]{P} \langle X, \tilde{X} \rangle_t$$

Corollary (IBP):

$$\int_0^t \tilde{X}_s \, dX_s = \tilde{X}_t X_t - \tilde{X}_0 X_0 - \int_0^t X_s \, d\tilde{X}_s - \langle X, \tilde{X} \rangle_t$$

(Itô correction)

Def (Fisk-Stratonovich integral)

$$\int_0^t \tilde{X}_s \circ dX_s := \int_0^t \tilde{X}_s \, dX_s + \frac{1}{2} \langle X, \tilde{X} \rangle_t .$$

Then both FTC & IBP again hold:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s$$

$$\int_0^t \tilde{X}_s \circ dX_s = \tilde{X}_t X_t - \tilde{X}_0 X_0 - \int_0^t X_s \circ d\tilde{X}_s$$

Thm (Levy characterization of SBM)

Let $B^{(1)}, \ldots, B^{(d)}$ be Brownian motions under same Brownian filtration

Let $Y^{(1)} \ldots Y^{(d)} \in \mathcal{V}^{loc}$ and let $\{X_t : t \geq 0\}$ be continuous with

$$\forall t \geq 0: \quad X_t = \sum_{k=1}^{d} \int_0^t Y_s^{(k)} \, dB_s^{(k)} \quad \text{a.s.}$$

Then $\forall t \geq 0: \langle X \rangle_t = t$

implies that $\{X_t : t \geq 0\}$ is a SBM.

Pf Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the common filtration.

Set $Z_t := e^{i\lambda X_t + \frac{\lambda^2}{2} \langle X \rangle_t}$

Itô:

$$dZ_t = i\lambda Z_t \, dX_t - \frac{\lambda^2}{2} Z_t \, d\langle X_t \rangle + \frac{\lambda^2}{2} Z_t \, d\langle X \rangle_t$$

$$= i\lambda Z_t \, dX_t$$

Since X is loc. martingale, so is Z

WTS $\{Z_t : t \geq 0\}$ is a proper martingale.

Lemma Let $\{M_s : s \geq 0\}$ be local martingale s.t.

$$\forall t \geq 0 : \quad \sup_{s \leq t} |M_s| \in L^\infty$$

Then $\{M_t : t \geq 0\}$ is a martingale

Pf: Let $\{T_n\}_{n \in \mathbb{N}}$ be stopping times s.t. $T_n \to \infty$ a.s and $\{M_{T_n \wedge t} : t \geq 0\}$ is a martingale. Then $\forall s \leq t$

$$E(M_{T_n \wedge t} | \mathcal{F}_s) = M_{T_n \wedge s}$$

BCT : $E(|M_{T_n \wedge t} - M_t|) \longrightarrow 0$ (assumption that $\sup_{s \leq t} |M_s| \in L^\infty$ needed here)

So $M_{T_n \wedge t} \longrightarrow M_t$ in L^1. This gives $E(M_{T_n \wedge t} | \mathcal{F}_s) \longrightarrow E(M_t | \mathcal{F}_s)$

Since Z is a martingale, we have ($\langle X \rangle_t = t$ used!)

$$E\left(e^{i\lambda X_t + \frac{\lambda^2}{2} t} \,\middle|\, \mathcal{F}_s\right) = e^{i\lambda X_s + \frac{\lambda^2}{2} s}$$

and so

$$E\left(e^{i\lambda(X_t - X_s)} \,\middle|\, \mathcal{F}_s\right) = e^{-\frac{\lambda^2}{2}(t-s)}.$$

So X has increments of SBM. Since X is continuous with $X_0 = 0$, it is SBM! $\square$