Itô formula for semimartingales

Last time: $dX_t = U_t\,dt + Y_t\,dB_t$. Then

$$\int_0^t Z_s\,dX_s := \int_0^t Z_s U_s\,ds + \int_0^t Z_s Y_s\,dB_s.$$

Thm For X = gen. diffusion, $f \in C^2(\mathbb{R})$, $t \geq 0$:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s.$$

$$df(X_t) = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,d\langle X\rangle_t.$$

Pf: Denote $A_t := X_0 + \int_0^t U_s\,ds$, $M_t := \int_0^t Y_s\,dB_s$. Then $X_t = A_t + M_t$.

STEP 1 (localization)

$$T_K := \inf\left\{ t \geq 0 : \begin{array}{l} |X_t| \geq K \vee |f'(X_t)| \geq K \vee |f''(X_t)| \geq K \\ \vee\, |M_t| \geq K \vee \int_0^t |U_s|\,ds \geq K \vee \int_0^t Y_s^2\,ds \geq K \end{array} \right\}$$

Then $T_K \xrightarrow[K\to\infty]{} \infty$ a.s.

Moreover: $X_{T_k \wedge t} = X_0 + \int_0^t U_s 1_{T_k > s} ds + \int_0^t Y_s 1_{T_k > s} dB_s$.

Once $T_k > t$, we can replace X by $X_{T_k \wedge \cdot}$ inside all formulas.

we may thus assume that:

$$\sup_{s \le t} |X_s| \le K \wedge \sup_{s \le t} |f'(X_s)| \le K \wedge \sup_{s \le t} |f''(X_s)| \le K$$

and $$\int_0^t |U_s| ds \le K \wedge \int_0^t Y_s^2 ds \le K \quad \text{a.s.}$$

STEP 2 (Taylor thm) Let $\Pi = \{0 = t_0 < \dots < t_n = t\}$

$$f(X_t) - f(X_0) = \sum_{i=1}^n (f(X_{t_i}) - f(X_{t_{i-1}}))$$

$$= \underbrace{\sum_{i=1}^n f'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})}_{J_t^{(1)}} + \underbrace{\frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})(\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}})}_{J_t^{(2)}}$$

$$+ \frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})\left[(M_{t_i} - M_{t_{i-1}})^2 - (\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}})\right] \leftarrow J_t^{(3)}$$

$$+ \frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})\left[(X_{t_i} - X_{t_{i-1}})^2 - (M_{t_i} - M_{t_{i-1}})^2\right] \leftarrow J_t^{(4)}$$

$$+ \sum_{i=1}^n \left(\int_0^1 \left[f''(\theta X_{t_{i-1}} + (1-\theta)X_{t_i}) - f''(X_{t_{i-1}})\right](1-\theta) d\theta\right)(X_{t_i} - X_{t_{i-1}})^2 \leftarrow J_t^{(5)}$$

$\boxed{\underline{\text{STEP 3}} \quad J_t^{(1)} \xrightarrow[\|\Pi\|\to 0]{P} \int_0^t f'(X_s)\,dX_s}$

$\boxed{\underline{\text{Lemma}}}$ For $u \le t$, W an $\mathcal{F}_u$-meas. RV: $W\int_u^t Y_s\,dB_s = \int_u^t W Y_s\,dB_s$

$$\left| J_t^{(1)} - \int_0^t f'(X_s)\,dX_s \right| \overset{\text{Lemma}}{=} \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} [f'(X_{t_{i-1}}) - f'(X_s)]\,dX_s \right|$$

$$\le \underbrace{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |f'(X_{t_{i-1}}) - f'(X_s)||U_s|\,ds}_{\le \operatorname{osc}_{f'\circ X}([0,t], \|\Pi\|)K} + \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f'(X_{t_{i-1}}) - f'(X_s)) Y_s\,dB_s \right|$$

$$E\left((2^{\text{nd}} \text{ term})^2 \right) \underset{+\text{Itô}}{\overset{\text{martingale}}{=}} \sum_{i=1}^n E\int_{t_{i-1}}^{t_i} |f'(X_{t_{i-1}}) - f'(X_s)|^2 Y_s^2\,ds \le E\, \underbrace{\operatorname{osc}_{f'\circ X}([0,t], \|\Pi\|)^2}_{\le (2K)^2} K \xrightarrow[\|\Pi\|\to 0]{\text{BCT}} 0$$

$\underline{\text{STEP 4}} \quad J_t^{(2)} \xrightarrow[\|\Pi\|\to 0]{} \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s$.

$\underline{\text{Pf:}}$ $f''\circ X$ is continuous, $s \mapsto \langle X\rangle_s$ is BV.

So Riemann sums converge to Stieltjes integral.

$\boxed{\underline{\text{STEP 5}} \quad J_t^{(4)} \xrightarrow[\|\Pi\|\to 0]{P} 0}$

Pf: $J_t^{(4)} = \underbrace{\frac{1}{2}\sum_{i=1}^{n} f''(X_{t_{i-1}})(A_{t_i} - A_{t_{i-1}})^2}_{1^{st}} + \underbrace{\sum_{i=1}^{n} f''(X_{t_{i-1}})(A_{t_i} - A_{t_{i-1}})(M_{t_i} - M_{t_{i-1}})}_{2^{nd}}$

$|1^{st}| \leq K\, V^{(2)}(A,\Pi) \leq K\, \underbrace{osc_A([0,t], \|\Pi\|)}_{\xrightarrow[\|\Pi\|\to 0]{} 0}\, \underbrace{V^{(1)}(A,\Pi)}_{\leq \int_0^t |U_s| ds \leq K}$

$|2^{nd}| \overset{CS}{\leq} K\, V^{(2)}(A,\Pi)^{1/2}\, V^{(2)}(M,\Pi)^{1/2} \xrightarrow[\|\Pi\|\to 0]{} 0$

$V^{(2)}(M,\Pi) \xrightarrow[\|\Pi\|\to 0]{P} \underbrace{\langle X \rangle_t}_{= \int_0^t Y_s^2 ds \leq K}$

$\boxed{\text{STEP 6} \quad J_t^{(5)} \xrightarrow[\|\Pi\|\to 0]{P} 0}$

Pf $|J_t^{(5)}| \leq \underbrace{osc_{f''\circ X}([0,t], \|\Pi\|)}_{\xrightarrow[\|\Pi\|\to 0]{} 0}\, V^{(2)}(X,\Pi) \longrightarrow \langle X \rangle_t \leq K$

$\boxed{\text{STEP 7} \quad J_t^{(3)} \xrightarrow[\|\Pi\|\to 0]{P} 0}$

Recall $J_t^{(3)} = \frac{1}{2}\sum_{i=1}^{n} f''(X_{t_{i-1}})\left[(M_{t_i}-M_{t_{i-1}})^2 - (\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}})\right]$

This a martingale (indexed by i) with bdd increments.

$$E\left((J_t^{(3)})^2\right) \le K^2\, E\left(\sum_{i=1}^{n}\left[(M_{t_i}-M_{t_{i-1}})^2 - (\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}})\right]^2\right)$$

$$\le 2K^2\, E\left(V_t^{(4)}(M,\Pi) + V_t^{(2)}(\langle X\rangle,\Pi)\right)$$

$$E\left(V_t^{(2)}(\langle X\rangle,\Pi)\right) \le E\left(\underbrace{\operatorname{osc}_{\langle X\rangle}([0,t],\|\Pi\|)}_{\le 2K,\ \to 0} K\right) \xrightarrow[\|\Pi\|\to 0]{\text{BCT}} 0$$

$$E\left(V_t^{(4)}(M,\Pi)\right) \le E\left(\underbrace{\operatorname{osc}_{M}([0,t],\|\Pi\|)^2}_{\le K,\ \xrightarrow[\|\Pi\|\to 0]{} 0} V^{(2)}(M,\Pi)\right)$$

Need: $\{V_t^{(2)}(M,\Pi) : \Pi\text{ - partition}\}$ is UI

Lemma Let M be a cont. martingale bounded by K.

Then $E\left(V_t^{(2)}(M,\Pi)^2\right) \le 48K^4 \quad (\forall\,\Pi\text{ - partition})$

<u>Pf of Lemma</u>:

$$E\left(V_t^{(2)}(M,\Pi)^2\right) = E\left(V_t^{(4)}(M,\Pi)\right) + 2\sum_{i=1}^{n} E\left((M_{t_i}-M_{t_{i-1}})^2 \sum_{j=i+1}^{n} (M_{t_j}-M_{t_{j-1}})^2\right)$$

$$E\left(V_t^{(4)}(M,\Pi)\right) \le (2K)^2 E\left(V_t^{(2)}(M,\Pi)\right) = 4K^2 E\left(\underbrace{(M_t-M_0)^2}_{\le (2K)^2}\right) \le 16K^4$$

$$2^{nd} \text{ term} = 2\sum_{i=1}^{n} E\left((M_{t_i}-M_{t_{i-1}})^2 (M_t - M_{t_i})^2\right) \le 2\cdot(2K)^2 E\left(\underbrace{V^{(2)}(M,\Pi)}_{\le 4K^2}\right) \le 32K^4$$

$\boxtimes$