Thm Let X take the form $dX_t = U_t\,dt + Y_t\,dB_t$.

Then $$V_t^{(2)}(X,\Pi) \xrightarrow[\|\Pi\|\to 0]{P} \int_0^t Y_s^2\,ds$$

Pf STEP 0 Reduce to $U_s = 0$.

STEP 1 ($Y \in V_0$) Assume $Y_s = \mathbb{1}_{\{0\}}(s)\, Z_0 + \sum_{i=1}^{m} Z_i\, \mathbb{1}_{(s_{i-1}, s_i]}(s)$

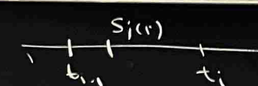
Assume $\Pi = \{0 = t_0 < \dots < t_k = t\}$ s.t. $\|\Pi\| < \min_{i=1\dots m} |s_i - s_{i-1}|$

Denote $I := \{ i = 1 \dots k : \{s_0, \dots, s_m\} \cap [t_{i-1}, t_i] \neq \emptyset \}$

$j(i) := \max\{ j = 1 \dots m : s_j \le t_i \}$

Then:

$i \in I \Rightarrow t_{i-1} \le s_{j(i)} \le t_i$



$i \notin I \Rightarrow t_{i-1} > s_{j(i)} \wedge s_{j(i)+1} > t_i$

So
$$V_t^{(2)}(X,\Pi) = \sum_{i=1}^{k} (X_{t_i} - X_{t_{i-1}})^2$$

1st

$$= \sum_{i\notin I} Z_{j(i)+1}^2 (B_{t_i} - B_{t_{i-1}})^2 + \sum_{i\in I} \left(Z_{j(i)} (B_{s_{j(i)}} - B_{t_{i-1}}) + Z_{j(i)+1} (B_{t_i} - B_{s_{j(i)}}) \right)^2$$

2nd

Denote $C := \max_{j=0..m} \|Z_j\|_\infty$.

$$2^{nd} \le 2C^2 \sum_{i\in I}\left[(B_{s(j)} - B_{t_{i-1}})^2 + (B_{t_i} - B_{s(j)})^2\right]$$

$$E(2^{nd}) \le 2C^2 \sum_{i\in I}(t_i - t_{i-1}) \le 2C^2 |I| \, \|\Pi\| \le 2C^2(m+1)\|\Pi\|$$

$$1^{st} - \underset{\sum_{i=1} Z^2_{j(i)+1}(t_i - t_{i-1})}{\underset{\|}{\int_0^t Y_s^2 \, ds}} = \underbrace{\sum_{i\notin I} Z^2_{j(i)+1}\left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right]}_{3^{rd}} - \underbrace{\sum_{i\in I} Z^2_{j(i)+1}(t_i - t_{i-1})}_{\le C^2(m+1)\|\Pi\|}$$

$$E((3^{rd})^2) \le C^4 \sum_{i\notin I} Var((B_{t_i} - B_{t_{i-1}})^2) = 3C^4 \sum_{i\notin I} |t_i - t_{i-1}|^2 \le 3C^4 t \|\Pi\|$$

So we get $V_t^{(2)}(X,\Pi) \xrightarrow[\|\Pi\|\to 0]{P} \int_0^t Y_s^2 \, ds$ for $Y \in V_0$.

STEP 2 $(Y \in V)$ Need:

$$\left| V_t^{(2)}(X,\Pi)^{1/2} - V_t^{(2)}(\bar{X},\Pi)^{1/2} \right|^2 \le V_t^{(2)}(X - \bar{X},\Pi)$$

Let $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ be s.t. $[\![Y - Y^{(n)}]\!] \xrightarrow[n\to\infty]{} 0$.

Denote $X_t^{(n)} := \int_0^t Y_s^{(n)}\, dB_s$. Then

$$E\left(\left| V_t^{(2)}(X,\Pi)^{1/2} - V_t^{(2)}(X^{(n)},\Pi)^{1/2} \right|^2 \right)$$

$$\leq E\, V_t^{(2)}(X - X^{(n)}, \Pi)$$

$$\overset{\text{martingale}}{=} E\left((X_t - X_t^{(n)})^2 \right)$$

$$\overset{\text{Itô}}{=} \int_0^t (Y_s - Y_s^{(n)})^2\, ds = \| Y - Y^{(n)} \|^2_{L^2([0,t]\times\Omega)} \xrightarrow[n\to\infty]{} 0$$

$$E\left(\left| \left(\int_0^t Y_s^2\, ds\right)^{1/2} - \left(\int_0^t (Y_s^{(n)})^2\, ds\right)^{1/2} \right|^2 \right) \downarrow$$

$$\leq E\left(\int_0^t (Y_s - Y_s^{(n)})^2\, ds \right) = \xrightarrow[n\to\infty]{} 0$$

Now apply 3ε argument under convergence in prob.

STEP 3 ($Y \in \mathcal{V}^{loc}$) Then $T^{(M)} := \inf\{t \ge 0 : \int_0^t Y_s^2 \, ds \ge M\}$.

on $T^{(M)} > t$ we have $X_u = X_{T^{(M)} \wedge u}$ $\quad u \le t$

and $X_{T^{(M)} \wedge t} = \int_0^t Y_s \mathbb{1}_{T^{(M)} > s} \, dB_s$ where $\{Y_s \mathbb{1}_{T^{(M)} > s} : s \ge 0\} \in \mathcal{V}$

so $$V_t^{(2)}(X_{T^{(M)} \wedge \cdot}, \Pi) \xrightarrow[\|\Pi\| \to 0]{P} \int_0^t Y_s^2 \mathbb{1}_{T^{(M)} > s} \, ds = \int_0^t Y_s^2 \, ds$$ on $\{T^{(M)} > t\}$

$$\| \quad V_t^{(2)}(X, \Pi)$$

$\boxtimes$

Def Given a generalized diffusion $dX_t = U_t \, dt + Y_t \, dB_t$, denote

$$\langle X \rangle_t := \int_0^t Y_s^2 \, ds$$

and call it the <u>quadratic variation process</u>.

Thm (Doob-Meyer decomposition for stoch integral)
Let X take the form $dX_t = Y_t\, dB_t$ for $Y \in \mathcal{V}_{loc}$.
Then $\{ X_t^2 - \langle X \rangle_t : t \geq 0 \}$ is a local martingale

Pf: ($Y \in \mathcal{V}$) Let $u \leq t$. Then

$$E(X_t^2 - X_u^2 \mid \mathcal{F}_u) = E((X_t - X_u)^2 \mid \mathcal{F}_u)$$

$$\underset{\text{Itô}}{\overset{\text{conditional}}{=}} E\left(\int_u^t Y_s^2\, ds \,\Big|\, \mathcal{F}_u \right) = E\left(\langle X \rangle_t - \langle X \rangle_u \mid \mathcal{F}_u \right)$$

So $E(X_t^2 - \langle X \rangle_t \mid \mathcal{F}_u) = X_u^2 - \langle X \rangle_u$ and the above is a (proper) martingale.

($Y \in \mathcal{V}_{loc}$) $\quad X_{T^{(M)} \wedge t} = \int_0^t Y_s 1_{T^{(M)} > s}\, dB_s$ where $\{ Y_s 1_{T^{(M)} > s} : s \geq 0 \} \in \mathcal{V}$

$$\langle X \rangle_{T^{(M)} \wedge t} = \int_0^t Y_s^2 1_{T^{(M)} > s}\, ds$$

So $\{ X^2_{T^{(M)} \wedge t} - \langle X \rangle_{T^{(M)} \wedge t} : t \geq 0 \}$ is a martingale.

Since $T^{(M)} \xrightarrow[M \to \infty]{} \infty$, $\{ X_t^2 - \langle X \rangle_t : t \geq 0 \}$ is <u>local</u> martingale. $\boxtimes$

Stochastic integral w.r.t. semimartingales

<u>Def</u> Given a generalized diffusion $dX_t = U_t\,dt + Y_t\,dB_t$ and a process $\{Z_t : t \geq 0\}$ that is adapted, jointly-measurable and obeys: $\forall t \geq 0$, $\int_0^t |U_s||Z_s|\,ds < \infty \wedge \int_0^t Z_s^2 Y_s^2\,ds < \infty$ a.s.

we set

$$\int_0^t Z_s\,dX_s := \int_0^t Z_s U_s\,ds + \int_0^t Z_s Y_s\,dB_s.$$

<u>Note</u>: This is well-defined and independent of representation of X using U and Y processes.

<u>Thm</u> Let X be generalized diffusion. Then $\forall f \in C^2(\mathbb{R})$ $\forall t \geq 0$:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s$$

(Stieltjes)

(generalized Itô formula)