Properties of Itô integral ($\mathcal{F}_0$ contains all P-null sets)

Thm Let $Y \in \mathcal{V}_{loc}$ and $T =$ stopping time. Then $\{Y_s \mathbb{1}_{T>s} ; s \geq 0\} \in \mathcal{V}_{loc}$ and $t \mapsto \int_0^t Y_s \, dB_s$ admits a continuous version s.t.

$$\int_0^{T \wedge t} Y_s \, dB_s = \int_0^t Y_s \mathbb{1}_{T>s} \, dB_s \quad \text{a.s.} \qquad (*)$$

In particular, if $\{T_N\}_{N \in \mathbb{N}}$ are stopping times s.t. $\forall N \in \mathbb{N}$: $\{Y_s \mathbb{1}_{T_N > s} ; s \geq 0\} \in \mathcal{V}$ and $T_N \to \infty$ a.s., then

$$\int_0^t Y_s \mathbb{1}_{T_N > s} \, dB_s \xrightarrow[N \to \infty]{P} \int_0^t Y_s \, dB_s$$

Pf: Recall $T^{(M)} = \inf\{t \geq 0 : \int_0^t Y_s^2 \, ds \geq M\}$. Then

$$\int_0^t Y_s \mathbb{1}_{T^{(M)} > s} \, dB_s = \int_0^t Y_s \, dB_s \quad \text{on } \{T^{(M)} > t\}$$

LHS admits a cont. version which extends to RHS.

$(*)$ follows from proposition and the fact: $\tilde{T}^{(M)} \wedge T = T^{(M)} \wedge T$

$$\tilde{T}^{(M)} := \inf\{t \geq 0 : \int_0^t Y_s^2 \mathbb{1}_{T>s} \, ds \geq M\}$$

$\int_0$

$$\int_0^{T \wedge t} Y_s \, dB_s \overset{\text{on } \{T^{(M)} > t\}}{=} \int_0^{T \wedge t} Y_s \mathbb{1}_{T^{(M)} > s} \, dB_s$$

$$\underset{\text{Prop}}{\overset{\text{a.s.}}{=}} \int_0^t Y_s \mathbb{1}_{T^{(M)} > s} \mathbb{1}_{T > s} \, dB_s$$

$$= \int_0^t Y_s \mathbb{1}_{T>s} \mathbb{1}_{\tilde{T}^{(M)} > s} \, dB_s \overset{\text{on } \{\tilde{T}^{(M)} > t\}}{=} \int_0^t Y_s \mathbb{1}_{T>s} \, dB_s \qquad \boxtimes$$

Lemma (Linearity) $\forall Y, \tilde{Y} \in V_{loc}$, $\forall \alpha, \beta \in \mathbb{R}$:

$$\alpha Y + \beta \tilde{Y} \in V_{loc}$$

and $\forall t \geq 0$:

$$\int_0^t (\alpha Y_s + \beta \tilde{Y}_s)\, dB_s = \alpha \int_0^t Y_s\, dB_s + \beta \int_0^t \tilde{Y}_s\, dB_s \quad \text{a.s.}$$

Pf $(\alpha Y_s + \beta \tilde{Y}_s)^2 \leq 2\alpha^2 Y_s^2 + 2\beta^2 \tilde{Y}_s^2$

Then linearity holds for $Y, \tilde{Y} \in V_0$ by explicit formula. L^2-limit extends to $Y, \tilde{Y} \in V$. For localization, use

$$T_N = \inf\{t \geq 0 : \int_0^t (Y_s^2 + \tilde{Y}_s^2)\, ds \geq M\}. \quad \boxtimes$$

Lemma (Additivity) Let $Y \in V_{loc}$ be defined relative to $\{B_t : t \geq 0\}$ and $\{\mathcal{F}_t\}_{t \geq 0}$.

Pick $u > 0$ and set

$$\tilde{B}_t := B_{t+u} - B_u, \quad \tilde{\mathcal{F}}_t := \mathcal{F}_{t+u}, \quad \tilde{Y}_s := Y_{u+s}.$$

Then $\tilde{B}$ is a Brownian motion, $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$ is a Brownian filtration for $\{\tilde{B}\}$ and $\tilde{Y} \in \tilde{V}^{loc}$ = loc. square integrable, adapted, jointly meas wrt $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$

and

$$\int_0^t \tilde{Y}_s\, d\tilde{B}_s = \int_0^{t+u} Y_s\, dB_s - \int_0^u Y_s\, dB_s \quad \text{a.s.}$$

Denote $\int_u^t Y_s \, dB_s := \int_0^t Y_s \, dB_s - \int_0^u Y_s \, dB_s$.

<u>Lemma</u> $\forall t \geq 0 \ \forall Y \in \mathcal{V}_{loc}$: $\int_0^t Y_s \, dB_s$ is $\mathcal{F}_t$-measurable.

if $Y \in \mathcal{V}$, then $\forall t \geq u \geq 0$:

$$E\left(\left(\int_u^t Y_s \, dB_s \right)^2 \Big| \mathcal{F}_u \right) = E\left(\int_0^t Y_s^2 \, ds \Big| \mathcal{F}_u \right)$$

(conditional Itô isometry).

<u>Lemma</u> $\forall Y, \tilde{Y} \in \mathcal{V} \ \forall t \geq 0$:

$$(*) \quad \int_0^t Y_s \, dB_s \overset{?}{=} \int_0^t \tilde{Y}_s \, dB_s \quad \text{a.s.}$$

is equivalent to

$$\mathrm{Leb}\left(\{ s \in [0,t] : Y_s \neq \tilde{Y}_s \} \right) = 0 \quad \text{a.s.} \quad (**)$$

Pf $* \Leftrightarrow E\left(\left(\int_0^t (Y_s - \tilde{Y}_s) \, dB_s \right)^2 \right) = 0 \Leftrightarrow \| Y - \tilde{Y} \|_{L^2([0,t] \times \Omega)} = 0 \overset{\text{Tonelli}}{\Leftrightarrow} (**)$ $\square$

Lemma Let $Y, \tilde{Y} \in \mathcal{V}_{loc}$. Then the events

$$\left\{ \omega \in \Omega : \left(\forall u \le t : \int_0^u Y_s^{(\omega)} dB_s^{(\omega)} = \int_0^u \tilde{Y}_s^{(\omega)} dB_s^{(\omega)} \right) \right\}$$

(assuming cont. versions)

and $\left\{ \omega \in \Omega : \mathrm{Leb}\left(\{ s \in [0,t] : Y_s(\omega) \neq \tilde{Y}_s(\omega) \} \right) = 0 \right\}$

differ only by P-null set.

Pf: HW.

Lemma Let $\{ Y^{(n)} \}_{n \in \mathbb{N}} \in (\mathcal{V}^{loc})^{\mathbb{N}}$ and $Y \in \mathcal{V}^{loc}$ be s.t.

$$\int_0^t (Y_s^{(n)} - Y_s)^2 ds \xrightarrow[n \to \infty]{P} 0.$$

Then $\int_0^t Y_s^{(n)} dB_s \xrightarrow[n \to \infty]{P} \int_0^t Y_s dB_s$

Itô formula:

Lemma Let $Y \in V^{loc}$ have continuous paths. Let $t \geq 0$.
Then for any sequence $\{\Pi_n\}$ of partitions s.t. $\Pi_n = \{0 = t_0^n < \dots < t_{m(n)}^n = t\}$

$$\sum_{i=1}^{m_n} Y_{t_{i-1}^n}\left(B_{t_i^n} - B_{t_{i-1}^n}\right) \xrightarrow[\|\Pi_n\| \to 0]{P} \int_0^t Y_s \, dB_s$$

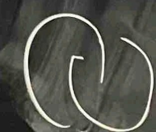
Pf: $Y_s^{(n)} = \sum_{i=1}^{m_n} Y_{t_{i-1}^n} 1_{(t_{i-1}^n, t_i]}(s)$, then LHS $= \int_0^t Y_s^{(n)} dB_s$.

Now $\int_0^t \left(Y_s^{(n)} - Y_s\right)^2 ds \leq osc_Y\left([0,t], \|\Pi_n\|\right)^2 t \xrightarrow[n \to \infty]{P} 0$ Now use above Lemma. $\boxtimes$

There is a 1-1 correspondence
btw $\{$contact str on M up to iso$\}$ $\sim$ $\{$open book of M up to positive stabilizations$\}$

unknot

$OB(A, \tau) = (S^3, \xi_{std})$



Hopf link