GOAL: Stoch. integral $\int_0^t Y_s \, dB_s$ for $Y \in \mathcal{V}_{loc}$.

$$\mathcal{V}_{loc} := \left\{ \{Y_t : t \geq 0\} : \begin{array}{l} \text{adapted, jointly measurable} \\ \text{and } \forall t \geq 0: \int_0^t Y_s^2 \, ds \underset{a.s.}{<} \infty \end{array} \right\}$$

Proposition: Let $Y \in \mathcal{V}$, T = stopping time. Then $\{Y_t 1_{T>t} : t \geq 0\} \in \mathcal{V}$.

and $\forall t \geq 0$:

$$\int_0^t Y_s 1_{T>s} \, dB_s \underset{a.s.}{=} \int_0^{T \wedge t} Y_s \, dB_s$$

continuous version evaluated at $T \wedge t$.

Lemma $T_N := 2^{-N} \lceil 2^N T \rceil$ is a stopping time $\forall N \geq 0$.
Moreover, $T_N - 2^{-N} \leq T \leq T_N$ and so $T_N \downarrow T$ as $N \to \infty$.
In particular, $\{Y_t 1_{T_N > t} : t \geq 0\} \in \mathcal{V}$ and $\{Y_t 1_{T>t} : t \geq 0\} \in \mathcal{V}$.

Pf Compared to last time, need to prove joint measurability.
Let U be RV. Then $U_n := 2^{-n} \lfloor 2^n U \rfloor$ obeys $U_n \uparrow U$.
Now $1_{U_n > t} = \sum_{k \in \mathbb{Z}} 1_{(k2^{-n}, \infty)}(t) \, 1_{k2^{-n} \leq U < (k+1)2^{-n}}$ is jointly meas.
But $U_n \uparrow U$ implies $1_{U_n > t} \uparrow 1_{U > t}$, so $\{1_{U>t} : t \geq 0\}$ is —— " —— $\boxtimes$

Now pick $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ s.t. $[\![Y^{(n)} - Y]\!] \xrightarrow[n\to\infty]{} 0$.

<u>Lemma</u>: $\forall n \in \mathbb{N}\ \forall N \geq 0$: $\{ Y_t^{(n)} 1_{T_N > t} : t \geq 0\} \in \mathcal{V}$ and

$$\int_0^t Y_s^{(n)} 1_{T_N > s}\, dB_s = \int_0^{t\wedge T_N} Y_s^{(n)}\, dB_s \quad \text{a.s.}$$

take explicit formula for simple processes and evaluate at $T_N \wedge t$

<u>Pf</u> Idea: Show that $\{ Y_s^{(n)} 1_{T_N > s} : s \geq 0\}$ is equivalent to a simple process. Then compare explicit formulas.

$$1_{T_N > t} = 1_{T > 2^{-N}\lfloor 2^N t \rfloor} = \sum_{k=0}^{\infty} 1_{T > k2^{-N}}\, 1_{[k2^{-N}, (k+1)2^{-N})}(t)$$

Assume $Y^{(n)}$ has representation $Y_t^{(n)} = Z_0 1_{\{0\}}(t) + \sum_{i=1}^{m} Z_i 1_{(t_{i-1}, t_i]}(t)$

where $t_m \in 2^{-N}\mathbb{N}$ and $\{k2^{-N} : k \geq 0 \wedge k2^{-N} \leq t_m\} \subseteq \{t_0, \ldots, t_m\}$.

Let $R_t := \sum_{i=1}^{m} Z_i 1_{T > t_{i-1}}\, 1_{(t_{i-1}, t_i]}(t)$.

Then $\left| Y_t^{(n)} 1_{T_N > t} - R_t \right| \leq \sum_{i=0}^{m} \|Z_i\|_\infty 1_{\{t_{i-1}\}}(t) \quad (t_{-1} := 0)$

So R is equivalent to $Y_\cdot^{(n)} 1_{T_N > \cdot}$ in $\mathcal{V}$.

$$\alpha := \int_0^t Y_s^{(n)} \mathbb{1}_{T_N > s}\, dB_s \overset{a.s.}{=} \int_0^t R_s\, dB_s$$

$$= \sum_{i=1}^m Z_i \mathbb{1}_{T > t_{i-1}} \left(B_{t\wedge t_i} - B_{t \wedge t_{i-1}}\right)$$

$$\beta := \int_0^{T_N \wedge t} Y_s^{(n)}\, dB_s = \sum_{i=1}^m Z_i \left(B_{T_N \wedge t \wedge t_i} - B_{T_N \wedge t \wedge t_{i-1}} \right)$$

So

$$|\alpha - \beta| \leq 2 \max_{i \leq m} \|Z_i\|_\infty \operatorname{osc}_B\left([0,t], \max_i |t_i - t_{i-1}|\right)$$

$$\xrightarrow[\max |t_i - t_{i-1}| \to 0]{} 0. \quad \boxtimes$$

<u>Now take limits:</u>

<u>Lemma</u> $\forall t \geq 0$: $\lim_{N\to\infty} \limsup_{n\to\infty} E\left(\left|\int_0^t Y_s^{(n)} \mathbb{1}_{T_N > s}\, dB_s - \int_0^t Y_s \mathbb{1}_{T > s}\, dB_s\right|^2\right) = 0$

Pf Expectation $\overset{\text{Itô}}{\leq} 2 \|Y^{(n)} - Y\|^2_{L^2([0,t]\times\Omega)} + 2E \int_0^t Y_s^2 \mathbb{1}_{T_N > s \geq T}\, ds$

$\xrightarrow[n\to\infty]{} 0$ $\qquad \mathbb{1}_{T_N > s \geq T} \longrightarrow \mathbb{1}_{\{T = s\}}$

$\xrightarrow[N\to\infty]{DCT} 0$

$\boxtimes$

Lemma $\int_0^{T_N \wedge t} Y_s^{(n)} dB_s \xrightarrow[n\to\infty]{P} \int_0^{T_N \wedge t} Y_s \, dB_s$.

Pf $P\left(|LHS - RHS| > \varepsilon\right) \leq P\left(\sup_{0 \leq u \leq t} \left|\int_0^u Y_s^{(n)} dB_s - \int_0^u Y_s \, dB_s\right| > \varepsilon\right)$

$\overset{\text{Doob-}L^2}{\leq} \frac{1}{\varepsilon^2} E\left(\left(\int_0^t (Y_s^{(n)} - Y_s) dB_s\right)^2\right)$

$\overset{\text{Itô}}{=} \frac{1}{\varepsilon^2} \| Y^{(n)} - Y \|^2_{L^2([0,t] \times \Omega)} \xrightarrow[n\to\infty]{} 0$. $\boxtimes$

Proof of Prop: From Lemmas:

$$\int_0^{T_N \wedge t} Y_s^{(n)} dB_s = \int_0^t Y_s^{(n)} \mathbb{1}_{T_N > s} dB_s \qquad \text{a.s.}$$

Now RHS $\xrightarrow[\substack{n\to\infty \\ N\to\infty}]{L^2} \int_0^t Y_s \mathbb{1}_{T>s} dB_s$, LHS $\xrightarrow[n\to\infty]{P} \int_0^{T_N \wedge t} Y_s dB_s$.

For continuous version of $u \longmapsto \int_0^u Y_s dB_s$; $\int_0^{T_N \wedge t} Y_s dB_s \longrightarrow \int_0^{T \wedge t} Y_s dB_s$.

So: $\int_0^t Y_s \mathbb{1}_{T>s} dB_s = \int_0^{t \wedge T} Y_s dB_s$ $\boxtimes$

Now if $Y \in \mathcal{V}_{loc}$, $T^{(M)} := \inf\{t \geq 0 : \int_0^t Y_s^2 ds \geq M\}$.

Then $\{Y_s 1_{T^{(M)} > s} : s \geq 0\} \in \mathcal{V}$.

Indeed: $\int_0^t Y_s^2 1_{T^{(M)} > s} ds \overset{MCT}{=} \lim_{u \uparrow T_M \wedge t} \int_0^u Y_s^2 ds \leq M$.

Now: for $N \geq M$:

$$\int_0^t Y_s 1_{T^{(M)} > s} dB_s = \int_0^t \underbrace{Y_s 1_{T^{(N)} > s}}_{\in \mathcal{V}} 1_{T^{(M)} > s} dB_s$$

$$\overset{Prop}{=} \int_0^{T_M \wedge t} Y_s 1_{T^{(N)} > s} dB_s = \int_0^t Y_s 1_{T^{(N)} > s} dB_s$$

on $\{T^{(M)} > t\}$.

So: $\int_0^t Y_s dB_s := \lim_{N \to \infty} \int_0^t Y_s 1_{T^{(N)} > s} dB_s$

exists a.s. on $\bigcup_{N \geq 1} \{T^{(N)} > t\}$, which is a full-measure set because $Y \in \mathcal{V}_{loc}$.

$\boxtimes$