Final stage of Itô integral

Last time: continuous version ; $\mathcal{F}_0$ contains all P-null sets

$\overline{\mathcal{V}_0}^{[\cdot]} = \mathcal{V}$.

Key lemma For $r_n(t,h) := 2^{-n}\lfloor 2^n(t-h)\rfloor + h$

$$\forall Y \in \mathcal{V}\ \forall T>0: \quad E\int_0^T dt \int_0^1 dh\, \left|Y_t - Y_{r_n(t,h)}\right|^2 \xrightarrow[n\to\infty]{} 0$$

Pf: Periodicity in h: $\int_0^1 f(r_n(t,h))\,dh = \frac{1}{2^{-n}}\int_0^{2^{-n}} f(t-h)\,dh$

When $\int_0^T Y_t(\omega)^2\,dt < \infty$, continuity of shift in $L^2(\mathbb{R})$ gives:

$$\int_0^T dt\, \left|Y_t(\omega) - Y_{t-h}(\omega)\right|^2 \xrightarrow[h\downarrow 0]{} 0$$

Integrate w.r.t. h to get

$$\int_0^T dt \int_0^1 dh\, \left|Y_t(\omega) - Y_{r_n(t,h)}\right|^2 = \frac{1}{2^{-n}}\int_0^{2^{-n}} dh \int_0^T dt\, \left|Y_t(\omega) - Y_{t-h}(\omega)\right|^2 \xrightarrow[n\to\infty]{\text{bound}} 0$$

DCT: True under expectation. $\boxtimes$

Today: Go beyond $Y \in \mathcal{V}$.

Def $\mathcal{V}_{loc} := \left\{ \{Y_t : t \in \mathbb{R}_+\} : \text{jointly meas. adapted and s.t. } \forall t \geq 0 : \int_0^t Y_s^2 \, ds < \infty \text{ a.s} \right\}$

Thm: Let $Y \in \mathcal{V}_{loc}$ and, for $M \geq 0$, set $T^{(M)} := \inf\{t \geq 0 : \int_0^t Y_s^2 \, ds \geq M\}$.
Then $\{Y_t \mathbb{1}_{T^{(M)} > t} : t \in \mathbb{R}_+\} \in \mathcal{V}$ and

$$\forall t \geq 0 \ \forall N \geq M : \int_0^t Y_s \mathbb{1}_{T^{(M)} > s} \, dB_s = \int_0^t Y_s \mathbb{1}_{T^{(N)} > s} \, dB_s \quad \text{a.s. on } \{T^{(M)} > t\}$$

In particular,

$$\forall Y \in \mathcal{V}_{loc} \ \forall t \geq 0 : \int_0^t Y_s \, dB_s := \lim_{N \to \infty} \int_0^t Y_s \mathbb{1}_{T^{(N)} > s} \, dB_s \quad \text{exists a.s.}$$

This will be our interpretation of $\int_0^t Y_s \, dB_s$ for $Y \in \mathcal{V}_{loc}$.

Def: Given filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$, an $\mathbb{R}_+ \cup \{\infty\}$-valued RV T is a __stopping time__ if

$$\forall t \in \mathbb{R}_+ : \{T \le t\} \in \mathcal{F}_t$$

Lemma $\forall Y \in \mathcal{V}_{loc}\ \forall M \ge 0 :\ T^{(M)}$ is a stopping time

Pf:
$$\{T^{(M)} \le t\} = \underbrace{\left\{\int_0^t Y_s^2\,ds \le M\right\}}_{\in \mathcal{F}_t} \cup \left(\underbrace{\left\{\int_0^t Y_s^2\,ds < M\right\}}_{\in \mathcal{F}_t} \cap \bigcap_{n \ge 1} \underbrace{\left\{\int_0^{t+1/n} Y_s^2\,ds = \infty\right\}}_{\in \mathcal{F}_0} \right).$$

Lemma $\forall Y \in \mathcal{V}_{loc}\ \forall M \ge 0 :\ \{Y_t \mathbb{1}_{T^{(M)} > t} : t \ge 0\} \in \mathcal{V}$

Pf $t \mapsto \mathbb{1}_{T^{(M)} > t}$ measurability — __To be resolved__

Adapted $\{\mathbb{1}_{T^{(M)} > t} = 1\} = \{T^{(M)} \le t\}^c$.

$$\int_0^t \left| Y_s \mathbb{1}_{T^{(M)} > s} \right|^2 ds \le tM \quad \text{a.s.} \qquad \boxed{X}\ (?)$$

Key proposition $\forall Y \in \mathcal{V}$ $\forall T$ stopping time:

$$\{Y_t \mathbb{1}_{T>0} : t \geq 0\} \in \mathcal{V}$$

and if $\{I_t : t \geq 0\}$ is a continuous version of $\int_0^\cdot Y_s \, dB_s$

then

$$\forall t \geq 0 : \int_0^{T \wedge t} Y_s \, dB_s := I_{T \wedge t} = \int_0^t Y_s \mathbb{1}_{T > s} \, dB_s \quad \text{a.s.}$$

Main idea: discretize T and Y simultaneously.

Lemma Given a stopping time T, let $T_N := 2^{-N} \lceil 2^N T \rceil$.

Then $\forall N \in \mathbb{N}$: T_N is a stopping time and $T_N - 2^{-N} \leq T \leq T_N$

and $T_N \downarrow T$ as $N \to \infty$

Pf:

$$\{T_N \leq t\} = \bigcup_{\substack{k \in \mathbb{Z} \\ k \leq t 2^N}} \{(k-1)2^{-N} < T \leq k 2^{-N}\} = \{T \leq \underbrace{2^{-N} \lfloor 2^N t \rfloor}_{\leq t}\} \in \mathcal{F}_t$$

Use of dyadics $\Rightarrow T_N \downarrow$. $\boxtimes$