Stochastic integral

last time: $\int_0^t Y_s \, dB_s$ $\quad Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$, $t \geq 0$,

today: $t \longmapsto \int_0^t Y_s \, dB_s$ admits a cont version on $\overline{\mathcal{V}_0}^{[\![\cdot]\!]} = \mathcal{V}$.

Def Given a filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ on $(\Omega, \mathcal{F}, P)$, we say $\{M_t : t \in \mathbb{R}_+\}$ is a martingale w.r.t. $\{\mathcal{F}_t\}$ if

1) $\forall t \in \mathbb{R}_+ : M_t \in L^1$ $\quad$ 2) $\forall t \geq 0 :$ M_t is $\mathcal{F}_t$-measurable

3) $\forall\, s \geq t \geq 0 : \quad E(M_s \mid \mathcal{F}_t) = M_t$ a.s.

If $M_t \in L^2 \, (\forall t)$ then we say M is L^2-martingale.

Text

Thm Let $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$. Then there exist a martingale $\{I_t : t \in \mathbb{R}_+\}$ s.t.

1) $\forall t \geq 0 : \; P\left(I_t = \int_0^t Y_s \, dB_s\right) = 1$

2) $t \longmapsto I_t$ is continuous.

In short, stoch integral admits a continuous version.

Here and henceforth we are assuming that $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets!!!

Lemma (Doob's L^2-ineq.) Let $\{M_t : t \in \mathbb{R}_+\}$ be a continuous L^2-martingale.

Then $\forall t \geq 0 \ \forall \lambda > 0$: $P\left(\sup_{s \in [0,t]} |M_s| > \lambda\right) \leq \frac{1}{\lambda^2} E(M_t^2)$

Pf Pick any $t_0, \dots, t_n \in [0,t]$ wlog $0 \leq t_0 < t_1 < \dots < t_n \leq t$.

$$P\left(\max_{k=1..n} |M_{t_k}| > \lambda\right) = \sum_{k=1}^{n} P\left(\max_{j<k} |M_{t_j}| \leq \lambda < |M_{t_k}|\right)$$

$$\overset{\text{Chebyshev}}{\leq} \frac{1}{\lambda^2} \sum_{k=1}^{n} E\left(\mathbb{1}_{\max_{j<k}|M_{t_j}| \leq \lambda < |M_{t_k}|} \, M_{t_k}^2\right)$$

$$\leq \frac{1}{\lambda^2} \sum_{k=1}^{n} E\left(\mathbb{1}_{\text{—\,''\,—}} \, M_t^2\right) \leq \frac{1}{\lambda^2} E(M_t^2)$$

$s \leq t$: $E(M_t^2 | \mathcal{F}_s) = E\left(M_s^2 + 2M_s(M_t - M_s) + \underbrace{(M_t - M_s)^2}_{\geq 0} \,\middle|\, \mathcal{F}_s\right) \geq M_s^2$

Now take $\{t_k\}_{k \in \mathbb{N}}$ s.t. $\bigcup_{k \in \mathbb{N}} \{t_k\} = \mathbb{Q} \cap [0,t]$.

MCT: $P\left(\sup_{s \in \mathbb{Q} \cap [0,t]} |M_s| > t\right) = \lim_{n \to \infty} P\left(\max_{k=1..n} |M_{t_k}| > \lambda\right) \leq \frac{E(M_t^2)}{\lambda^2}$,

By continuity, same holds for $\sup_{s \leq t} |M_s|$. ☒

Now let $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$ and $\{Y_n\}_{n\in\mathbb{N}} \in \mathcal{V}_0$ s.t.

$$[\![Y - Y_n]\!] \leq 2^{-n}$$

Set $I_t^{(n)} = \int_0^t Y_s^{(n)} \, dB_s$

Lemma $\{I_t^{(n)} : t \in \mathbb{R}_+\}$ is a continuous L^2-martingale

Pf. $$Y_t^{(n)} = Z_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{m_n} Z_i \mathbb{1}_{(t_{i-1}, t_i]}(t)$$

$0 = t_0 < \ldots < t_{m_n}$ $\quad Z_0 \ldots Z_{m_n} \in L^2$

$$I_t^{(n)} = \sum_{i=1}^{m_n} Z_i (B_{t\wedge t_i} - B_{t \wedge t_{i-1}})$$

Then B-continuous $\Rightarrow I^{(n)}$ continuous.

Adapted & integrable clear from definition.

$$E\left(Z_i \left(B_{(t+s)\wedge t_i} - B_{(t+s)\wedge t_{i-1}}\right) \middle| \mathcal{F}_t \right)$$

$$= Z_i \left(B_{t\wedge t_i} - B_{t\wedge t_{i-1}} \right)$$

$\boxtimes$

$$P\Big(\sup_{s\le t}|I^{(n)}_s - I^{(n+1)}_s| > 2^{-n/2}\Big)$$

$$\overset{\text{Doob}}{\le} \frac{1}{2^{-n}} E\big(|I^{(n)}_t - I^{(n+1)}_t|^2\big)$$

$$\overset{\text{Itô}}{=} \frac{1}{2^{-n}} E\int_0^t (Y^{(n)}_s - Y^{(n+1)}_s)^2\, ds$$

$$= \frac{1}{2^{-n}} \| Y^{(n)} - Y^{(n+1)}\|^2_{L^2([0,t]\times\Omega)}$$

$$\le \frac{2^{\lceil t\rceil}}{2^{-n}} \big[Y^{(n)} - Y^{(n+1)}\big]^2 \le C_t 2^{-n}.$$

So this prob. is summable on n.

Now: $$I_t = \begin{cases} \lim_{n\to\infty} I^{(n)}_t & \text{on } \bigcap_{N\ge 1} \Big\{ \sup_{s\le N} |I^{(n)}_s - I^{(n+1)}_s| > 2^{-n/2} \text{ i.o.}\Big\}^c \\ 0 & \text{else} \end{cases}$$

So construction ensures that $t \to I_t$ is continuous.

Since $I^{(n)}_t \xrightarrow{L^2} \int_0^t Y_s\, ds$, we have $I_t = \int_0^t Y_s\, dB_s$ a.s. (t).

If $\mathcal{F}_0$ contains all P-null sets, then I is adapted.

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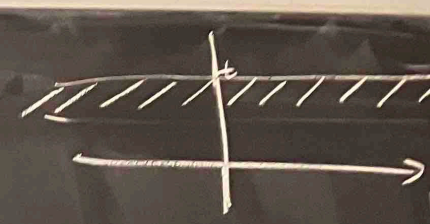
Thm $\boxed{\overline{\mathcal{V}_0}^{[\![\cdot]\!]} = \mathcal{V}}$.



Pf: $r_n(t,h) := 2^{-n}\lfloor 2^n(t-h)\rfloor + h$.

$t - 2^{-n} \le r_n(t,h) \le t$, 2^{-n}-periodic in h.

so $$\int_0^1 f(r_n(t,h))\,dh = 2^n \int_0^{2^{-n}} f(t-h)\,dh.$$ (f Borel, bdd)

Pick $Y \in \mathcal{V}$.

Key Lemma: $$E\int_0^T dt \int_0^1 dh\, |Y_t - Y_{r_n(t,h)}|^2 \xrightarrow[n\to\infty]{} 0.$$

Cantor diag argument: $\exists \{n_k\}_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ $n_k \to \infty$ s.t.

$$\forall k\in\mathbb{N}: \quad E\Big(\int_0^k dt \int_0^1 dh\, |Y_t - Y_{r_{n_k}(t,h)}|^2 \le 4^{-k}$$

Proof of lemma: see the notes

Markov + Fubini:

$$\mathrm{Leb}\left(h \in [0,1] : \; \mathbb{E} \int_0^k dt \, |Y_t - Y_{r_{n_k}(t,h)}|^2 > 2^{-k} \right) \le 2^{-k}$$

BC: $\exists h \in [0,1]$ s.t. $\mathbb{E} \int_0^k dt \, |Y_t - Y_{r_{n_k}(t,h)}|^2 \xrightarrow[k\to\infty]{} 0$

So $[\![Y - Y_{r_{n_k}(\cdot,h)}]\!] \longrightarrow 0$. and $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$.

<u>Note</u> Example of "probabilistic method".