MATH 275D take-home final exam

due at or before 11:59PM on 3/28/2024

Instructions: Solve the following problems. You may use results from class, provided you provide a full and correct statement thereof. The exam is open book but you must work alone and not discuss the exam with other students. Return in PDF format, preferably typeset in LateX, by above due date via email to biskup@math.ucla.edu.

Problem 1: Given a standard Brownian motion $\{B_t \colon t \geqslant 0\}$ started at zero and a stopping time τ of a Brownian filtration, prove that

$$E(\tau) < \infty \;\Rightarrow\; E(B_\tau^2) = E(\tau) \;\wedge\; E(B_\tau) = 0$$

Note: The same holds even when just $E(\tau^{1/2}) < \infty$.

Problem 2: Let $f \in C(\mathbb{R})$ be such that, for some $g, h \in C(\mathbb{R})$ and with $\{B_t \colon t \geqslant 0\}$ denoting the standard Brownian motion started from zero,

$$f(B_t) = f(B_0) + \int_0^t g(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t h(B_s)\,\mathrm{d}s$$

holds a.s. for each $t \geqslant 0$. Do the following:

(1) prove that $f \in C^2(\mathbb{R})$, and
(2) conclude that $g = f'$ and $h = f''$.

Problem 3: Let M_t be a local Itô martingale; i.e., $M_t = \int_0^t Y_s\,\mathrm{d}B_s$ for a jointly-measurable process Y adapted to a Brownian filtration, defined via localization for all $t < \tau_\infty := \lim_{n\to\infty} \tau_n$ where

$$\tau_n := \inf\Bigl\{t \geqslant 0 \colon \int_0^t Y_s^2\,\mathrm{d}s \geqslant n\Bigr\}.$$

Assuming that $\int_0^\infty Y_s^2\,\mathrm{d}s = \infty$ a.s., prove that

$$\limsup_{t\uparrow\tau_\infty} M_t = +\infty \;\wedge\; \liminf_{t\uparrow\tau_\infty} M_t = -\infty$$

a.s. on $\{\tau_\infty < \infty\}$. In particular, $t \mapsto M_t$ cannot be extended continuously to $t = \tau_\infty$.

Problem 4: Let $a < b$ be reals and let $\sigma \colon (a,b) \to \mathbb{R} \smallsetminus \{0\}$ a Borel function such that σ^{-2} is integrable on (a,b). This ensures existence and pathwise uniqueness of a strong solution X to the SDE

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t,$$

up to the first exit time from (a,b). Denote $\tau_v := \inf\{t \geqslant 0 \colon X_t = v\}$ and write P^x for the law of X with $P^x(X_0 = x) = 1$ with expectation denoted E^x. Do as follows:

(1) Prove that $P^x(\tau_a \wedge \tau_b < \infty) = 1$ for all $x \in (a,b)$.
(2) Compute $P^x(\tau_a < \tau_b)$ as a function of x.
(3) Compute $E^x(\tau_a \wedge \tau_b)$ as a function of x.

Hint: Consider suitable martingales. Answers in form of integrals over $\mathbb{R}$ are permitted.