9. Continuous version and Itô integrability

The purpose of this lecture is to resolve two natural questions. The first one concerns a characterization of processes that we can actually integrate in the above sense. Thanks to Lemma 8.8 we know that all left-continuous processes $Y \in \mathcal{V}$ are included, but the point is to determine how large is the class of integrable processes in $\mathcal{V}$. The second question concerns the t-dependence of the integral $\int_0^t Y_s \mathrm{d}B_s$. Note that we constructed the integral for each t separately, and only modulo a null set, but if are to treat $\{\int_0^t Y_s \mathrm{d}B_s \colon t \in [0,\infty)\}$ as a process then we need to establish some minimal regularity of the integrals as a function of their upper limit.

We start by addressing the second question. For this we recall the following definition:

Definition 9.1 *Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be a filtration. A process $\{M_t \colon t \geqslant 0\}$ is a martingale if it is adapted (i.e., $\forall t \geqslant 0\colon\ M_t$ is $\mathcal{F}_t$-measurable), integrable (meaning $\forall t \geqslant 0\colon\ M_t \in L^1$) and*

$$\forall t,s \geqslant 0\colon \quad E(M_{t+s}|\mathcal{F}_t) = M_t \text{ a.s.} \tag{9.1}$$

We say that the martingale is continuous if $t \mapsto M_t$ is continuous. A martingale is an L^2-martingale if $\forall t \geqslant 0\colon\ M_t \in L^2$.

We then have:

Theorem 9.2 *Suppose the Brownian filtration is such that $\mathcal{F}_0$ contains all P-null sets. For each $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$ there exists a continuous L^2-martingale $\{I_t \colon t \geqslant 0\}$ such that*

$$\forall t \geqslant 0\colon \quad P\Bigl(I_t = \int_0^t Y_s \,\mathrm{d}B_s\Bigr) = 1 \tag{9.2}$$

In particular, the process $\{\int_0^t Y_s \mathrm{d}B_s \colon t \geqslant 0\}$ of stochastic integrals admits a continuous version.

For the proof we need the following result from martingale theory:

Lemma 9.3 (Doob's L^2-inequality) *Let $\{M_t \colon t \geqslant 0\}$ be a continuous L^2-martingale. Then*

$$\forall t \geqslant 0\, \forall \lambda > 0\colon \quad P\Bigl(\sup_{s\in[0,t]} |M_s| > \lambda\Bigr) \leqslant \frac{1}{\lambda^2} E(M_t^2) \tag{9.3}$$

Proof. Fix $t \geqslant 0$ and $\lambda > 0$. We will use the fact that the statement is known for discrete-time martingales. Indeed, given any set of points $\{t_i \colon i = 0,\dots,n\} \subseteq [0,t]$, which we may assume to obey $0 = t_0 < t_1 < \dots < t_n \leqslant t$, the process $\{M_{t_i}\}_{i=1}^n$ is an L^2-martingale. The discrete-time Doob L^2-inequality then reads

$$P\Bigl(\max_{i=0,\dots,n} |M_{t_i}| > \lambda\Bigr) \leqslant \frac{1}{\lambda^2} E(M_{t_n}^2) \leqslant \frac{1}{\lambda^2} E(M_t^2), \tag{9.4}$$

where we also used that $t \mapsto E(M_t^2)$ is non-decreasing. Refining the family of points to increase to all rationals in $[0,t]$ yields

$$P\Bigl(\sup_{s\in[0,t]\cap\mathbb{Q}} |M_s| > \lambda\Bigr) \leqslant \frac{1}{\lambda^2} E(M_t^2) \tag{9.5}$$

where the Monotone Convergence Theorem was used on the left-hand side to pass the limit inside the probability. As M has continuous paths, the supremum in (9.5) equals that in (9.3) and so we are done. $\square$

Proof of Theorem 9.2. Since $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$, for each $n \geqslant 1$ there is $Y^{(n)} \in \mathcal{V}_0$ such that

$$[\![Y^{(n)} - Y]\!] \leqslant 2^{-n}. \tag{9.6}$$

Since $[\![\widetilde{Y}]\!] \geqslant 2^{-k} \min\{1, \|\widetilde{Y}\|_{L^2([0,k]\times\Omega)}\}$, for $n \geqslant \lceil t \rceil$ this implies

$$\|Y^{(n)} - Y\|_{L^2([0,t]\times\Omega)} \leqslant 2^{-n+\lceil t \rceil} \tag{9.7}$$

Define

$$I_t^{(n)} := \int_0^t Y_s^{(n)} \, \mathrm{d}B_s. \tag{9.8}$$

Then $t \mapsto I_t^{(n)}$ is continuous with $I_t^{(n)} \in L^2$ and $\mathcal{F}_t$-measurable. We claim:

Lemma 9.4 *$I^{(n)}$ is a martingale.*

Proof. In light of the above, it remains to check the martingale property (9.1). Assuming $Y^{(n)}$ is given by the right-hand side of (8.5), this boils down to showing that, for any $t_i > t_{i-1} \geqslant 0$, any $t, s \geqslant 0$, any $Z_i \in L^2$ that is $\mathcal{F}_{t_{i-1}}$-measurable,

$$E\big(Z_i(B_{t_i\wedge(t+s)} - B_{t_{i-1}\wedge(t+s)}) \,\big|\, \mathcal{F}_t\big) = Z_i(B_{t_i\wedge t} - B_{t_{i-1}\wedge t}) \tag{9.9}$$

For $t \geqslant t_i$ this is immediate from the fact that all of the random variables on the left are $\mathcal{F}_{t_i}$-measurable, while for $t \leqslant t_{i-1}$ we first condition on $\mathcal{F}_{t_{i-1}}$ and then realize that this conditional expectation vanishes, as does the right-hand side. For $t \in (t_{i-1}, t_i)$ we can move Z_i out of the conditional expectation. Then we write the Brownian increment on the left as

$$(B_{t_i\wedge(t+s)} - B_t) + (B_t - B_{t_{i-1}}) \tag{9.10}$$

The conditional expectation (given $\mathcal{F}_t$) of the first term in the parentheses vanishes thanks to Definition 8.1(3), while the second term is $\mathcal{F}_t$-measurable and thus unaffected by taking this conditional expectation. Noting that, for this range of t, also the increment on the right equals $B_t - B_{t_{i-1}}$, we get (9.9). $\square$

Returning to the proof of Theorem 9.2, we now observe that Doob's L^2-martingale inequality along with Itô isometry and (9.7) show, for any $m \geqslant n$, that

$$\begin{aligned} P\Big(\sup_{s\in[0,t]} |I_s^{(n)} - I_s^{(m)}| > 2^{-n/2}\Big) &\leqslant 2^n E\big[(I_t^{(n)} - I_t^{(m)})^2\big] \\ &= 2^n \|Y^{(n)} - Y^{(m)}\|^2_{L^2([0,t]\times\Omega)} \leqslant 4 \cdot 2^{-m\wedge n+\lceil t \rceil} \end{aligned} \tag{9.11}$$

For $m := n+1$ this is summable on n and so, by the Borel-Cantelli lemma, the event on the left occurs only finitely often a.s.

To extract the desired conclusion from this, we set

$$I_t := \begin{cases} \lim\limits_{n\to\infty} I_t^{(n)} & \text{on } \bigcap\limits_{N\geqslant 1}\Big\{ \sup\limits_{s\in[0,N]} |I_s^{(n)} - I_s^{(n+1)}| > 2^{-n/2} \text{ i.o.}(n)\Big\}^{\mathrm{c}} \\ 0 & \text{else,} \end{cases} \tag{9.12}$$

and note that the limit exist on the event in the first alternative, which by the previous reasoning is a full measure event.

The limit in (9.12) is actually locally uniform, since each $I^{(n)}$ is continuous, also the limit process I is continuous. The construction of the Itô integral gives $I_t^{(n)} \to \int_0^t Y_s \mathrm{d}B_s$ in L^2 and so we get (9.2) as well as $I_t \in L^2$. Since $\mathcal{F}_0$ contains all P-null sets, and thus also the set on which the second alternative in (9.12) applies, the process I is adapted. Passing in the limit in the defining property (9.1), Lemma 9.4 implies that I is a continuous L^2-martingale as claimed. □

As to the other question discussed above, here we simply claim:

Theorem 9.5 *Using the setting and notation defined above,*

$$\overline{\mathcal{V}_0}^{[\![\cdot]\!]} = \mathcal{V} \tag{9.13}$$

In particular, $\int_0^t Y \mathrm{d}B_s$ is defined as in Corollary 8.7 for all $Y \in \mathcal{V}$ and all $t \geqslant 0$.

Proof. Let $Y \in \mathcal{V}$. We need to show that Y can be approximated by a sequence of processes from $\mathcal{V}_0$. In Lemma 8.8 we used the values of Y at dyadic rationals to give a good approximation for Y left-continuous. Here we will use a similar idea but for dyadic rationals shifted by a random value so that a.e. shift actually produces a good approximation. The proof is thus an example of a *probabilistic method* which is a way to demonstrate existence of an object by showing that a sample from a natural probability measure yields a desired object with probability one.

Moving to the actual argument, for technical reasons we first extend $t \mapsto Y_t$ to all reals by setting $Y_t = 0$ for all $t < 0$. Define, for each integer $n \geqslant 1$ and reals $t, h \geqslant 0$,

$$r_n(t,h) := 2^{-n}\lfloor 2^n(t-h)\rfloor + h \tag{9.14}$$

Then $t \mapsto r_n(t,h)$ takes only values in $2^{-n}\mathbb{Z} + h$ and obeys

$$t - 2^{-n} \leqslant r_n(t,h) \leqslant t \tag{9.15}$$

Note also that $h \mapsto r_n(t,h)$ is a piecewise linear (but discontinuous) periodic function with period 2^{-n}. In particular, for each integer $n \geqslant 1$,

$$\int_0^1 f\big(r_n(t,h)\big)\mathrm{d}h = 2^n \int_0^{2^{-n}} f(t-h)\mathrm{d}h \tag{9.16}$$

holds for each Borel $f\colon \mathbb{R} \to [0,\infty)$ and each $t \geqslant 0$. This identity drives the proof of:

Lemma 9.6 *For all $T > 0$,*

$$E\int_0^T \mathrm{d}t \int_0^1 \mathrm{d}h\, |Y_t - Y_{r_n(t,h)}|^2 \underset{n\to\infty}{\longrightarrow} 0 \tag{9.17}$$

Proof. Fix $T > 0$. We first claim that, whenever $\int_0^T Y_t^s \mathrm{d}t < \infty$, we have

$$\int_0^T |Y_t - Y_{t-h}|^2 \,\mathrm{d}t \underset{h\downarrow 0}{\longrightarrow} 0 \tag{9.18}$$

This holds because, for each $\omega \in \Omega$, the map $t \mapsto Y_t(\omega)$ — being measurable and square integrable on $[0,T]$ thanks to $Y \in \mathcal{V}$ — can be approximated in $L^2([0,t])$ by a sequence of bounded continuous functions for which the claim holds by the Bounded Convergence Theorem. (This argument is generally used to prove that the shift-map is continuous in L^p for every $p \in [1,\infty)$.)

Since the quantity in (9.18) is bounded by $4\int_0^T Y_t^s \mathrm{d}t$, which has finite expectation by $Y \in \mathcal{V}$, the Dominated Convergence Theorem yields

$$E\int_0^T |Y_t - Y_{t-h}|^2 \,\mathrm{d}t \underset{h\downarrow 0}{\longrightarrow} 0 \tag{9.19}$$

Noting that (9.16) along with Fubini-Tonelli gives

$$E\int_0^T \mathrm{d}t \int_0^1 \mathrm{d}h\, |Y_t - Y_{r_n(t,h)}|^2 = 2^n \int_0^{2^{-n}} \Bigl(E\int_0^T |Y_t - Y_{t-h}|^2 \,\mathrm{d}t\Bigr)\mathrm{d}h, \tag{9.20}$$

the claim follows by plugging (9.19) on the right-hand side. $\square$

Moving back to the main line of the proof of Theorem 9.5, using Lemma 9.6 along with the Cantor diagonal argument we now find an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}\colon\ E\int_0^k \mathrm{d}t \int_0^1 \mathrm{d}h\, |Y_t - Y_{r_{n_k}(t,h)}|^2 \leqslant 4^{-k} \tag{9.21}$$

The Markov inequality along with Fubini-Tonelli implies

$$\lambda\biggl(\Bigl\{h \in [0,1]\colon E\int_0^k \mathrm{d}t\, |Y_t - Y_{r_{n_k}(t,h)}|^2 > 2^{-k}\Bigr\}\biggr) \leqslant 2^{-k}, \tag{9.22}$$

where λ is the Lebesgue measure on $[0,1]$. The Borel-Cantelli lemma now implies that, for Lebesgue-a.e $h \in [0,1]$,

$$E\int_0^m \mathrm{d}t\, |Y_t - Y_{r_{n_k}(t,h)}|^2 \underset{k\to\infty}{\longrightarrow} 0 \tag{9.23}$$

holds for all $m \geqslant 1$.

In order to conclude the proof, pick any h in the full-measure set such that (9.23) holds. The discrete nature of $r_n(t,h)$ then implies that $Y_{r_{n_k}(t,h)} \in \mathcal{V}_0$ and (9.23) gives $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$. This proves the claim. $\square$

Further reading: Section 3.1 in Øksendal, Section 3.2B in Karatzas-Shreve