5. Properties of Brownian paths

Having settled the question of uniqueness of Brownian motion, we can now explore the properties of Brownian paths.

5.1 Brownian symmetries.

Our first observation concerns transformations of Brownian paths that produce again Brownian paths.

Proposition 5.1 (Brownian symmetries) *Let $\{B_t\colon t \in [0,\infty)\}$ be a standard Brownian motion. Then so is the process $\{W_t\colon t \in [0,\infty)\}$ where, for all $t \geqslant 0$,*

(1) $W_t := B_{t+s} - B_s$, *for a fixed* $s \geqslant 0$,
(2) $W_t := \frac{1}{a} B_{a^2 t}$, *for a fixed* $a \neq 0$,
(3) *W defined by*

$$W_t := \begin{cases} tB_{1/t}, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \end{cases} \tag{5.1}$$

on the event $\{\lim_{t\downarrow 0} tB_{1/t} = 0\}$, and by $W_t := 0$ on the complement thereof.

We call the transformation $B \to W$ as follows: (1) is a shift, (2) is a diffusive scaling and (3) is the time inversion.

Proof. As to (1-2), it is easy to check that W has independent increments with the correct law. Since the continuity of B implies the continuity of W, the claim follows.

For (3), let $\widetilde{W}_t$ be defined by the dichotomy in (5.1). First we claim that $\{\widetilde{W}_t\colon t \in [0,\infty)\}$ has the finite-dimensional distributions of standard Brownian motion. For this we need:

Lemma 5.2 *The conditions (2-3) in Definition 1.6 are equivalent to*

(2') $\{B_t\colon t \in [0,\infty)\}$ *is (multivariate) Gaussian,*
(3') $\forall t,s \geqslant 0\colon\ E(B_t) = 0 \ \wedge\ E(B_sB_t) = s \wedge t$

Leaving the easy proof of this to homework, we now observe that $\widetilde{W}$ is (multivariate) Gaussian with mean zero and

$$\forall t,s > 0\colon \quad E\big(\widetilde{W}_s\widetilde{W}_t\big) = ts\Big(\frac{1}{t} \wedge \frac{1}{s}\Big) = t \wedge s. \tag{5.2}$$

Since $\widetilde{W}_0 = 0$, it follows that $\{\widetilde{W}_t\colon t \in [0,\infty)\}$ satisfies (1-3) of Definition 1.6. Next we note that $t \mapsto \widetilde{W}_t$ is continuous on $(0,\infty)$. This implies

$$P\big(\lim_{t\downarrow 0} \widetilde{W}_t = 0\big) = P\bigg(\bigcup_{n\geqslant 1}\bigcup_{m\geqslant 1}\bigcap_{k\geqslant m}\Big\{\sup_{t\in\mathbb{Q}\cap(0,1/k)} |\widetilde{W}_t| < 1/n\Big\}\bigg) \tag{5.3}$$

But the above equality of finite dimensional distributions implies that $\{\widetilde{W}_t\colon t \in \mathbb{Q} \cap (0,\infty)\}$ is equidistributed to $\{B_t\colon t \in \mathbb{Q} \cap (0,\infty)\}$, and so

$$P\Big(\sup_{t\in\mathbb{Q}\cap(0,1/k)} |\widetilde{W}_t| < 1/n\Big) = P\Big(\sup_{t\in\mathbb{Q}\cap(0,1/k)} |B_t| < 1/n\Big). \tag{5.4}$$

Using this in (5.3), we conclude

$$P\big(\lim_{t\downarrow 0}\widetilde{W}_t = 0\big) = P\big(\lim_{t\downarrow 0} B_t = 0\big) = 1, \tag{5.5}$$

where we also used continuity of $t \mapsto B_t$ on $[0,\infty)$. It follows that the second alternative in the definition of W occurs with zero probability and so W is indistinguishable from $\widetilde{W}$. Having continuous paths, W is a standard Brownian motion, as desired. ☐

One argument in the proof is worthy of separate claim:

Corollary 5.3 (SLLN for standard Brownian motion) *For a standard Brownian motion* $\{B_t\colon t\in[0,\infty)\}$,

$$\lim_{t\to\infty}\frac{B_t}{t} = 0 \quad \text{a.s.} \tag{5.6}$$

Proof. The claim is equivalent to $\lim_{t\downarrow 0} tB_{1/t} = 0$ which was shown to be a full measure event in the previous proof. ☐

Note that the SLLN implies (5.6) for t restricted to naturals. The claim (5.6) could then be proved by noting that

$$\lim_{n\to\infty}\frac{1}{n}\sup_{n\leqslant t\leqslant n+1}|B_t - B_n| = 0 \quad \text{a.s.} \tag{5.7}$$

because, as needs to be checked the suprema are i.i.d. having a Gaussian tail (and thus the first moment). The existence of the second moment allows us to infer (5.7) even with $\frac{1}{n}$ replaced by $\frac{1}{\sqrt{n}}$, but that scaling does not quite work for(5.6), as we will see in Corollary 5.6 below.

We remark that additional symmetries exist for the *d-dimensional Brownian motion*, which is an $\mathbb{R}^d$-valued process with Cartesian coordinates

$$\vec{B}_t := \big(B_t^{(1)},\dots,B_t^{(d)}\big) \tag{5.8}$$

where $B^{(i)}\colon i = 1,\dots,d\}$ are independent standard Brownian motions. For instance, given any $d\times d$-orthogonal matrix U, the process $U\vec{B}_t$ is again a d-dimensional Brownian motion. In fact, the law of the Brownian motion is preserved even by the conformal transforms of $\mathbb{R}^d$, provided one is willing to re-parametrize time.

5.2 Sample path regularity.

As a next item of interest, we will review the known fact about the regularity of Brownian sample paths. In Corollary 4.7 we noted that a.e. Brownian path is locally-uniformly γ-Hölder for every $\gamma < 1/2$. To complement this we note:

Theorem 5.4 (Paley, Wiener and Zygmund 1933) *Let $\gamma > 1/2$. Then for a.e. path $\{B_t\colon t\in[0,\infty)\}$ of standard Brownian motion,*

$$\forall t\geqslant 0\colon\qquad \limsup_{s\downarrow t}\frac{|B_t - B_s|}{|t-s|^\gamma} = \infty \tag{5.9}$$

In short, a.e. Brownian path is nowhere γ-Hölder and, in particular, nowhere differentiable.

Proof. Fix $\gamma \in (1/2,1)$ and let ℓ be an integer satisfying $\ell(\gamma-1/2)>1$. If $t\in[0,1/2]$ is such that the *limes superior* in (5.9) is finite, then for some $C\in(0,\infty)$ and $\delta_0\in(0,1/2)$,

$$\forall\delta\in(0,\delta_0):\quad |B_{t+\delta}-B_t|\leqslant C\delta^\gamma \tag{5.10}$$

For all integer $n\geqslant 1$ satisfying $(\ell+1)2^{-n}<\delta_0$, all $j=1,\dots,\ell$ and $k:=\lceil 2^n t\rceil$ we then have

$$\begin{aligned}|B_{(k+j)2^{-n}}-B_{(k+j-1)2^{-n}}| &\leqslant |B_{(k+j)2^{-n}}-B_t|+|B_{(k+j-1)2^{-n}}-B_t|\\ &\leqslant 2C[(\ell+1)2^{-n}]^\gamma = 2C(\ell+1)^\gamma\, 2^{-n\gamma},\end{aligned} \tag{5.11}$$

where we used that $(k+j)2^{-n}-t\leqslant(\ell+1)2^{-n}$. Denoting, for integers $m,n,k\geqslant 1$,

$$A^{(m)}_{n,k,j}:=\bigl\{|B_{(k+j)2^{-n}}-B_{(k+j-1)2^{-n}}|\leqslant m2^{-n\gamma}\bigr\} \tag{5.12}$$

we have thus proved

$$\Bigl\{\exists t\in[0,1/2]\colon\ \limsup_{s\downarrow t}\frac{|B_t-B_s|}{|t-s|^\gamma}<\infty\Bigr\}\subseteq\bigcup_{m\geqslant 1}\Bigl\{\bigcup_{k=1}^{2^n}\bigcap_{j=1}^{\ell}A^{(m)}_{n,k,j}\ \text{ i.o.}(n)\Bigr\} \tag{5.13}$$

where "i.o.(n)" means "for infinitely many n."

We will now show that the event on the right-hand side of (5.13) has vanishing probability. For this note that, using that $\mathcal{N}(0,\sigma^2)=\sigma\mathcal{N}(0,1)$,

$$\begin{aligned}P\bigl(A^{(m)}_{n,k,j}\bigr) &= P\bigl(|\mathcal{N}(0,2^{-n})|\leqslant m2^{-n\gamma}\bigr)\\ &= P\bigl(|\mathcal{N}(0,1)|\leqslant m2^{-n(\gamma-1/2)}\bigr)\leqslant 2m2^{-n(\gamma-1/2)}\end{aligned} \tag{5.14}$$

where the inequality relies on the fact that the probability density of $\mathcal{N}(0,1)$ is everywhere less than one. The stationarity and independence of Brownian increments along with the union bound then give

$$P\Bigl(\bigcup_{k=1}^{2^n}\bigcap_{j=1}^{\ell}A^{(m)}_{n,k,j}\Bigr)\leqslant 2^n\bigl[2m2^{-n(\gamma-1/2)}\bigr]^\ell=(2m)^\gamma 2^{-n[\ell(\gamma-1/2)-1]} \tag{5.15}$$

By our choice of ℓ, the right-hand side is summable on n and so the Borel-Cantelli lemma shows that the probability that this event occurs for infinitely many n is zero. In light of the inclusion (5.13) this proves the claim for $t\in[0,1/2]$. Using the stationarity of increments, the conclusion extends readily to all $t\geqslant 0$. $\square$

We remark that the previous proof provides a blueprint of many other regularity results for Brownian paths. Indeed, we are trying to prove/disprove a property of paths at a *continuum* of t's. Covering the favorable set of paths by a discrete collection of events based, quite typically, on dyadic rationals, we achieve countability which then permits to complete the calculation using the union bound, the Borel-Cantelli lemma and some problem-specific calculations. This proof-strategy often causes the statement to contain a certain inexplicit caveat: Instead of worrying first about whether the event on the left of (5.13) is measurable (is it not, actually?), we rather assert (and prove) that its defining property does not occur on a measurable set of full measure.

Notice that Theorem 5.4 can be interpreted as saying that a "typical" random path is nowhere differentiable, where the word "typical" is in the sense of measure (more precisely, the Wiener measure). Another way to express "typicality" is via the notion of *category*; indeed, a standard application of the Baire Category Theorem is that the set of nowhere differentiable functions is co-meager (a.k.a. second category or "large" in the sense of category) among all continuous functions on $[0,1]$. Each of these results provides a proof of existence of nowhere differentiable functions, a result that was considered a challenge in certain days (that is, until Weierstrass' example from 1878).

Combining Corollary 4.7 with Theorem 5.4 we conclude that $\gamma = 1/2 + o(1)$ is the correct Hölder exponent of Brownian paths. A question is: Can the $o(1)$ correction be controlled or refined? The answer is in the affirmative, but the proofs are lengthy so we only contend ourselves with the statements of the salient conclusions.

We start with the behavior at a typical point on a Brownian path:

Theorem 5.5 (Law of Iterated Logarithm, Khinchin 1933) *Let $\{B_t\colon t \in [0,\infty)\}$ be a standard Brownian motion. Then for all $t \geqslant 0$,*

$$\limsup_{s\downarrow t} \frac{|B_t - B_s|}{\sqrt{|t-s|\log\log\frac{1}{|t-s|}}} = \sqrt{2} \quad \text{a.s} \tag{5.16}$$

We emphasize that the implicit null set in (5.16) depends on t. That this represents behavior at "typical" points we note (or prove in homework) that the set of $t \geqslant 0$ for which equality holds is a Borel set of full Lebesgue measure.

Thanks to the Brownian shift symmetry, it suffices to prove (5.16) at $t = 0$. The time inversion symmetry then converts this to a statement about the maximal growth rate of Brownian path at large times:

Corollary 5.6 *Let $\{B_t\colon t \in [0,\infty)\}$ be a standard Brownian motion. Then*

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{t\log\log t}} = \sqrt{2} \quad \text{a.s.} \tag{5.17}$$

In this formulation, the conclusion extends also to suitably normalized random walks — more precisely, those is the domain of attraction of the Brownian motion.

While Khinchin's Law of the Iterated Logarithm desribes the maximal path oscillation at typical times, it does not capture the worst-case regimes. This is the content of the next two theorems:

Theorem 5.7 (Lévy modulus of continuity, Lévy 1937) *Let $\{B_t\colon t \in [0,\infty)\}$ be a standard Brownian motion. Then*

$$\limsup_{\delta\downarrow 0} \sup_{\substack{t,s\geqslant 0\\ 0<|t-s|<\delta}} \frac{|B_t - B_s|}{\sqrt{\delta\log(1/\delta)}} = \sqrt{2} \quad \text{a.s.} \tag{5.18}$$

Theorem 5.8 (Dvoretzky 1963, Davis 1983) *Let $\{B_t\colon t \in [0,\infty)\}$ be a standard Brownian motion. Then*

$$\inf_{t\geqslant 0} \limsup_{s\downarrow t} \frac{|B_t - B_s|}{\sqrt{|t-s|}} = 1 \quad \text{a.s.} \tag{5.19}$$

In conclusion, a.e. Brownian path contains points at which the oscillation over interval of size δ grows proportionally to $\sqrt{\delta \log(1/\delta)}$ — we call these the *fast points* — as well as points where the path oscillates only proportionally to $\sqrt{\delta}$ — we call these the *slow points*. Even more is known; for instance, the Hausdorff dimension of the set of fast points with proportionality constant at least $\lambda \in (0, \sqrt{2})$. Similarly for the set of slow points where the oscillation grows at least as $a\sqrt{\delta}$, for any $a \geqslant 1$. The book by P. Mörters and Y. Peres discusses these, along with detailed proofs.

Further reading: Section 2.9 of Karatzas-Shreve