4. Wiener space and Wiener measure

Here we will address the question of uniqueness of stochastic processes with continuous sample paths and, in particular, the standard Brownian motion. First note that, synthesizing our earlier developments, we are finally able to claim:

Theorem 4.1 *A standard Brownian motion exists.*

Proof. The Kolmogorov Extension Theorem implies existence of a probability space $(\Omega, \mathcal{F}, P)$ supporting a family of random variables $\{B_t\colon t \in [0,\infty)\}$ obeying 2-3 of Definition 1.6. The fact that

$$\mathcal{N}(0,\sigma^2) = \sigma\mathcal{N}(0,1) \tag{4.1}$$

then shows, for any $a > 0$ and any $t, s \geqslant 0$,

$$E\big(|B_t - B_s|^a\big) = |t-s|^{a/2} E\big(|\mathcal{N}(0,1)^a|\big) \tag{4.2}$$

Since a normal random variable has all moments, the Kolmogorov-Čenstov continuity condition holds for any $a > 2$ with $b := a/2 - 1$. By Theorem 3.4, B admits a continuous version $\widetilde{B}$, which then obeys 2-4 of Definition 1.6. As $B_0 = 0$ a.s., another modification ensures that $\widetilde{B}_0 = 0$. In particular, $\{\widetilde{B}_t\colon t \in [0,\infty)\}$ is a standard Brownian motion. □

With this in hand, our next question is: Can other constructions be put forward that produce a standard Brownian motion that is distinct from ours? This is not an irrelevant consideration because there are, in fact, a number of ways to construct a stochastic process satisfying Definition 1.6. Here is a list:

(1) a construction in the proof of Theorem 4.1 based on the Kolmogorov Extension Theorem and the Kolmogorov-Čenstov continuity criterion,
(2) the *Levy construction* which represents a path of Brownian motion as the limit of a random infinite series,
(3) a construction based on Donsker's Theorem, which asserts that any random walk with centered step distribution of unit variance (and steps interpolated piecewise linearly) tends in law (in the sense to be discussed) to standard Brownian motion,
(4) a construction in Folland's "Analysis" based on the Riesz representation theorem applied to the space $\overline{\mathbb{R}}^{[0,\infty)}$, for $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, endowed with product topology.

An interesting aspect of (4) is, in spite of working in a product space, it yields a continuous process directly. This is because the class of Borel sets $\mathcal{B}(\overline{\mathbb{R}}^{[0,\infty)})$ associated with the product topology contains the set of continuous functions, thus avoiding the "countability curse" that plagues (1). (Note that $\mathcal{B}(\mathbb{R})^{\otimes[0,\infty)} \subsetneq \mathcal{B}(\overline{\mathbb{R}}^{[0,\infty)})$.)

Each of the above constructions produces a family of random variables $\{B_t\colon t \in [0,\infty)\}$ that induces, through the standard procedure, the same measure on the infinite product space

$$\big(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes[0,\infty)}\big). \tag{4.3}$$

But this measure is equally induced by the discontinuous process $\widetilde{B}$ obtained by chaining B_t to B_t+1 for t equal to the value of an independent exponential; cf (2.36). It appears that continuity is a necessary condition for uniqueness, if there is uniqueness at all.

This spawns an idea: Let us view the standard Brownian motion as the continuous-function space-valued random variable; i.e., one taking values in

$$C[0,\infty) := \bigl\{f\in\mathbb{R}^{[0,\infty)}\colon\ \text{continuous}\bigr\} \tag{4.4}$$

We endow this set with the metric

$$\rho(f,g) := \sum_{n\geqslant 1} 2^{-n} \sup_{t,s\in[0,n]} \bigl|f(t)-g(t)\bigr| \wedge 1 \tag{4.5}$$

inducing the topology of locally-uniform convergence. The reason for requiring just local uniformity (instead of its global counterpart) is to ensure:

Lemma 4.2 *The metric space $(C[0,\infty),\rho)$ is complete and separable. In particular, there are $\{f_n\}_{n\geqslant1}\in C[0,\infty)$ such that every open set in $C[0,\infty)$ is a countable union of finite intersections of sets $\{U_k(f_n,a)\colon k,n\geqslant 1,\ a\in\mathbb{Q}^+\}$ where*

$$U_k(f,a) := \Bigl\{g\in C[0,\infty)\colon \sup_{t\in[0,k]} |g(t)-f(t)|<a\Bigr\}. \tag{4.6}$$

We leave the proof of this statement to homework. Note that the claim implies that

$$\bigl(C[0,\infty),\mathcal{B}(C[0,\infty))\bigr) \tag{4.7}$$

is a standard Borel space. (This is useful in the construction via Donsker's Theorem mentioned earlier, as that involves weak convergence for which complete and separable metric spaces are a perfect setting.) We call (4.7) the *canonical space* or the *Wiener space*, after N. Wiener. Next we observe:

Lemma 4.3 *Let $\{X_t\colon t\in[0,\infty)\}$ be an $\mathbb{R}$-valued stochastic process whose every sample path is continuous. Then $t\mapsto X_t$ is a $C[0,\infty)$-valued random variable.*

Proof. Let $(\Omega,\mathcal{F})$ be a measurable space on which $X=\{X_t\colon t\in[0,\infty)\}$ are realized. Using the notation (4.6) and continuity of X, we have

$$X^{-1}\bigl(U_k(f,a)\bigr) = \bigcup_{a'\in\mathbb{Q}\cap[0,a)} \ \bigcap_{t\in[0,k]\cap\mathbb{Q}} \bigl\{|X_t-f(t)|<a'\bigr\} \tag{4.8}$$

Each set in the countable union/intersection belongs to $\mathcal{F}$ and thus so does $X^{-1}(U_k(f,a))$. Noting that, for any map $X\colon\Omega\to\mathscr{X}$ and any σ-algebra Σ on $\mathscr{X}$,

$$\bigl\{A\in\Sigma\colon X^{-1}(A)\in\mathcal{F}\bigr\} \text{ is a } \sigma\text{-algebra}, \tag{4.9}$$

the claim follows from Lemma 4.2. □

An important consequence of the previous lemma is:

Corollary 4.4 *An $\mathbb{R}$-valued stochastic process $\{X_t\colon t\in[0,\infty)\}$ on $(\Omega,\mathcal{F},P)$ with continuous paths induces a unique measure $P^{(X)}$ on $(C[0,\infty),\mathcal{B}(C[0,\infty)))$ by*

$$\forall A\in\mathcal{B}(C[0,\infty))\colon\quad P^{(X)}(A) := P(X\in A) \tag{4.10}$$

Our final question is what reasonable properties determine a probability measure on the Wiener space. This is the content of:

Lemma 4.5 *Let $\mathcal{P} \subseteq \mathcal{B}(\mathbb{R})$ be a π-system that generates $\mathcal{B}(\mathbb{R})$. Every probability measure μ on the Wiener space is determined by its finitely-dimensional marginals; i.e., the restriction of μ to sets of the form*

$$\bigcap_{i=1}^{n} \{\omega \in C[0,\infty) \colon \omega(t_i) \in A_i\} \tag{4.11}$$

over all $n \geqslant 1$, all $t_1, \dots, t_n \in [0,\infty)$ and all $A_1, \dots, A_n \in \mathcal{P}$.

We again leave the proof of this to homework. As a consequence we get:

Corollary 4.6 *Every realization of standard Brownian motion induces (via (4.10)) the same distribution on the Wiener space.*

Proof. Any two standard Brownian motions have the same finite dimensional marginals and so their induced measures agree on the sets of the form (4.11), for $\mathcal{P}$ being, say, the set of all open intervals in $\mathbb{R}$. These generate $\mathcal{B}(\mathbb{R})$ and Lemma 4.5 thus forces the induced measures to be equal. □

Corollary 4.6 expresses the sense in which the standard Brownian motion is unique. We will call the measure it induces on the Wiener space the *Wiener measure*. A "practical" aspect of having uniqueness is that whatever we can prove about one version of standard Brownian motion, holds for all other versions provided it can be expressed using a measurable subset of the Wiener space. Here is an example:

Corollary 4.7 *Every standard Brownian motion $B = \{B_t \colon t \in [0,\infty)\}$ obeys: For all $\gamma \in (0,1/2)$ and a.e. B,*

$$\forall t_0 \geqslant 0\colon \quad \sup_{0\leqslant s<t\leqslant t_0} \frac{|B_s - B_t|}{|t-s|^\gamma} < \infty. \tag{4.12}$$

In short, a.e. Brownian path is locally γ-Hölder continuous for every $\gamma \in (0,1/2)$.

Proof. Recall that in the proof of Theorem 4.1 we verified the Kolmogorov-Čenstov condition, and thus proved local γ-Hölder continuity for $\gamma \in (0,1/2)$, for every $a > 2$ and $b := \frac{a}{2} - 1$. As $b/a = \frac{1}{2} - \frac{1}{a}$, taking $a \to \infty$ the claim follows for the construction provided via the Kolmogorov-Čenstov Theorem. (Note that we have to take $a \to \infty$ along a sequence as each application potentially discards a new null set.)

To see that the same applies to all ways to realize standard Brownian motion, we need to show that it applies to the sample from the Wiener measure (and $B_t(\omega) = \omega(t)$). For this it suffices to show that the supremum in (4.12) is a measurable as a map from $C[0,\infty)$ to the extended reals. This follows by the fact that, thanks to the continuity, the supremum can be restricted to rational s and t along with the fact that $\omega \mapsto B_t(\omega)$ is continuous and thus measurable for each $t \geqslant 0$. □

Further reading: Section 2.4 of Karatzas-Shreve