3. Continuity criteria

Having discussed tools for proving existence of stochastic processes with given distribution, here we will address the conditions under which that stochastic processes admits a continuous version. We start with some general observations.

3.1 Countability curse and version of a process.

Looking back at the definition of the standard Brownian motion, the Kolmogorov Extension Theorem gives us a family of random variables $\{B_t \colon t \in [0,\infty)\}$ with prescribed finite dimensional distributions; i.e., (1-3) of Definition 1.6. However, there seems to be no information concerning part (4); namely, the continuity of $t \mapsto B_t$.

As we will now explain, this question in fact cannot be resolved in the framework of product measure spaces. Recall that, given a family of random variables $\{X_t \colon t \in T\}$ taking values in measurable space $(\mathscr{X}, \Sigma)$,

$$\sigma(X_t \colon t \in T) := \sigma\Big(\big\{\{X_t \in A\} \colon t \in T,\, A \in \Sigma\big\}\Big) \tag{3.1}$$

is the smallest σ-algebra in the underlying probability space that makes all of these measurable. Next note the following general fact:

Lemma 3.1 (Countability curse) *For any stochastic process $\{X_t \colon t \in T\}$,*

$$\sigma(X_t \colon t \in T) = \bigcup_{\substack{S \subseteq T \\ \text{countable}}} \sigma(X_t \colon t \in S) \tag{3.2}$$

This statement, whose easy proof is left to homework, implies that any event measurable with respect to $\sigma(X_t \colon t \in T)$ is determined by at most a countable number of X_t's. Applying this in the context of product spaces, we get:

Corollary 3.2 *Let $\{X_t \colon t \in [0,\infty)\}$ be an $\mathbb{R}$-valued stochastic process realized via canonical coordinate projections on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes[0,\infty)})$. Then*

$$\big\{t \mapsto X_t \text{ is continuous}\big\} \notin \sigma\big(X_t \colon t \in [0,\infty)\big) \tag{3.3}$$

In fact, for any $A \in \sigma(X_t \colon t \in [0,\infty))$ we have

$$A \subseteq \big\{t \mapsto X_t \text{ is continuous}\big\} \;\Rightarrow\; A = \varnothing \tag{3.4}$$

The product σ-algebra $\mathcal{F}$ that arises in the proof of the Kolmogorov Extension Theorem coincides with $\sigma(X_t \colon t \in T)$ and so (3.3) shows that the continuity of sample paths $t \mapsto X_t$ cannot even be *asked* within the product space framework.

The reason for adding the clauses (3.4) to the statement is that one might perhaps hope that the event "B is continuous" is equivalent to a measurable set — meaning that it can be written as the union of a measurable set and a subset of a P-null set. Then (3.4) would force $\{B \text{ is continuous}\}$ to be a null set under the completion $\bar{P}$ of P, which would be quite discouraging for our endeavors. However, as we will note in Lemma 3.5, for Brownian motion constructed via the Kolmogorov Extension Theorem this alternative does not occur and the set $\{B \text{ is continuous}\}$ is not measurable even under $\bar{P}$.

The problem arises from the uncountable nature of the underlying index set. In order to present a solution, we turn back to the example (2.36) which shows that a continuous process over an uncountable index set can be changed, or *modified*, to make it discontinuous without altering its finite-dimensional distributions. This leads to:

Definition 3.3 *Given two processes $X = \{X_t : t \in T\}$ and $Y = \{Y_t : t \in T\}$ on the same probability space, we say that Y is a* version *(or a* modification*) of X if*

$$\forall t \in T: \quad P(X_t = Y_t) = 1 \tag{3.5}$$

As is easy to check, all versions of a given process have the same finite dimensional distributions. Clearly, when T is countable, two versions are equal everywhere with probability one, and so they are *indistinguishable* thus making the concept of a version uninteresting. As shown in (2.36), this is quite different for T uncountable.

3.2 Kolmogorov-Čenstov Theorem.

The point of the above remarks is that we can turn the apparent deficiency of uncountable index sets to our favor by asking: What conditions on the finite dimensional distributions ensure the existence of a continuous version? A natural guess is to require suitable continuity/regularity of the law of $|X_t - X_s|$ in the limit as $s \to t$. This leads to criteria an early example of which is:

Theorem 3.4 (Kolmogorov-Čenstov) *Let $X = \{X_t : t \in [0, \infty)\}$ be an $\mathbb{R}$-valued stochastic process such that*

$$\exists C, a, b > 0\, \forall t, s \geqslant 0: \quad E\big(|X_t - X_s|^a\big) \leqslant C|t-s|^{1+b} \tag{3.6}$$

Then X admits a continuous version $\widetilde{X}$ that obeys

$$\forall \gamma \in (0, b/a)\, \forall t_0 > 0: \sup_{0 \leqslant s < t \leqslant t_0} \frac{|\widetilde{X}_t - \widetilde{X}_s|}{|t-s|^\gamma} < \infty. \tag{3.7}$$

In short, $\widetilde{X}$ is locally γ-Hölder for every $\gamma \in (0, b/a)$.

Proof. We start with some simple estimates. Let a, b, C be as in (3.7) and pick $\gamma \in (0, b/a)$. The Markov inequality shows

$$P\big(|X_t - X_s| > \lambda\big) \leqslant \frac{1}{\lambda^a} E\big(|X_t - X_s|^a\big) \leqslant \frac{C}{\lambda^a}|t-s|^{1+b} \tag{3.8}$$

Hence we get, for each integer $n \geqslant 1$,

$$P\big(|X_{t+2^{-n}} - X_t| > 2^{-\gamma n}\big) \leqslant C 2^{-n} 2^{-n(b-\gamma a)} \tag{3.9}$$

The union bound then gives

$$\begin{aligned} &P\Big(\max_{k=1,\dots,2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| > 2^{-\gamma n}\Big) \\ &\qquad \leqslant \sum_{k=1}^{2^n} P\big(|X_{k2^{-n}} - X_{(k-1)2^{-n}}| > 2^{-\gamma n}\big) \leqslant C \sum_{k=1}^{2^n} 2^{-n} 2^{-n(b-\gamma a)} = C 2^{-n(b-\gamma a)} \end{aligned} \tag{3.10}$$

Thanks to $b > \gamma a$, this is summable on $n \geqslant 1$. The Borel-Cantelli lemma now yields an integer-valued random variable n_0 with

$$P(n_0 < \infty) = 1 \tag{3.11}$$

and such that

$$\forall n \geqslant n_0 \, \forall k = 1, \dots, 2^n \colon \quad |X_{k2^{-n}} - X_{(k-1)2^{-n}}| \leqslant 2^{-\gamma n}. \tag{3.12}$$

We will now show how this statement implies the existence of a version of X satisfying (3.8) for $t_0 := 1$.

Denote

$$\mathbb{D}_n := \{k2^{-n} \colon k = 0, \dots, 2^{-n}\} \tag{3.13}$$

and observe that if $s, t \in \mathbb{D}_n$ obey $s < t$, then there exist $s', t' \in \mathbb{D}_{n-1}$ such that $s \leqslant s' \leqslant t' \leqslant t$ and $|s - s'| \leqslant 2^{-n}$ and $|t - t'| \leqslant 2^{-n}$. For these points we get

$$|X_t - X_s| \leqslant |X_{t'} - X_{s'}| + |X_t - X_{t'}| + |X_s - X_{s'}|. \tag{3.14}$$

Given any $\delta > 0$, (3.13) then shows

$$\forall n \geqslant n_0 \colon \quad \max_{\substack{s,t \in \mathbb{D}_n \\ |s-t| < \delta}} |X_t - X_s| \leqslant 2 \cdot 2^{-\gamma n} + \max_{\substack{s,t \in \mathbb{D}_{n-1} \\ |s-t| < \delta}} |X_t - X_s|. \tag{3.15}$$

The maximum on the right will be trivially zero if $\delta < 2^{-n+1}$. Iterating (3.16) we thus get

$$\forall \delta \in (0, 2^{-n_0}) \colon \quad \sup_{t,s \in \bigcup_{n > n_0} \mathbb{D}_n} |X_t - X_s| \leqslant 2 \sum_{n \colon 2^{-n} \leqslant \delta} 2^{-\gamma n} \leqslant \frac{2}{1 - 2^{-\gamma}} \, \delta^\gamma \tag{3.16}$$

It follows that, on $\{n_0 < \infty\}$, the process $t \mapsto X_t$ restricted to $\mathbb{D} := \bigcup_{n \geqslant 1} \mathbb{D}_n$ is γ-Hölder continuous. In particular, the limit in

$$\widetilde{X}_t := \begin{cases} \lim\limits_{s \to t, \, s \in \mathbb{D}} X_t, & \text{on } \{n_0 < \infty\}, \\ X_0, & \text{on } \{n_0 = \infty\}, \end{cases} \tag{3.17}$$

exists and the resulting process obeys (3.8) with $t_0 := 1$. Finally, note that, by (3.7) again,

$$P\big(|X_t - X_s| > \epsilon\big) \leqslant \frac{C}{\epsilon^a} |t - s|^{1+b} \underset{s \to t}{\longrightarrow} 0 \tag{3.18}$$

Applying (3.18) along with Fatou's lemma then shows

$$\forall \epsilon > 0 \colon \quad P\big(|\widetilde{X}_t - X_t| > \epsilon\big) = 0 \tag{3.19}$$

and so $\widetilde{X}$ is indeed a version of X on $[0, 1]$.

It remains to extend the above construction to $[0, \infty)$. For this we decompose the positive reals into the union $\bigcup_{n \geqslant 1} (\frac{n}{2} + [0, 1])$. In each of the intervals on the right we have a version $\widetilde{X}$ of the process that obeys the corresponding formulation of (3.8). In light of (3.18), these versions coincide with one another on the overlap of this interval a.s. and so they extend to a version on all of $[0, \infty)$. $\square$

3.3 Remarks.

We proceed with some remarks. First we note the following elementary consequence of the existence of a continuous version of a process on a product space:

Lemma 3.5 *Let $\{X_t \colon t \in [0,\infty)\}$ be an $\mathbb{R}$-valued stochastic process realized on the product space $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes[0,\infty)})$ via coordinate projections, $X_t(\omega) := \omega(t)$. Assume that X admits a continuous version. Then for any $A \in \sigma(X_t \colon t \in [0,\infty))$,*

$$\big\{t \mapsto X_t \text{ is continuous}\big\} \subseteq A \;\Rightarrow\; P(A) = 1. \tag{3.20}$$

Proof. By Lemma 3.1, there is a countable $S \subseteq [0,\infty)$ such that $A \in \sigma(X_t \colon t \in S)$. We may assume without loss of generality that S is dense in $[0,\infty)$. Note that the premise of the claim reads

$$A^{\mathrm{c}} \subseteq \big\{t \mapsto X_t \text{ is NOT continuous}\big\} \tag{3.21}$$

If $\omega \in A^{\mathrm{c}}$ is such that $t \mapsto \omega(t)$ is uniformly continuous on $S \cap [0,n]$ for each $n \geqslant 1$, then $\tilde\omega_t(t) := \lim_{s\downarrow t,\, s\in S} \omega(t)$ exists and equals $\omega(t)$ for each $t \in S$. Since $\omega \in A^{\mathrm{c}}$ and $A^{\mathrm{c}} \in \mathcal{F}_S$ imply that any modification of ω outside S also belongs to A^{c}, we get $\tilde\omega \in A^{\mathrm{c}}$. But $\tilde\omega$ is continuous, in contradiction with (3.21).

It follows that

$$A^{\mathrm{c}} \subseteq \bigcup_{n\geqslant 1} \big\{t \mapsto X_t \text{ is NOT uniformly continuous on } S\big\} \tag{3.22}$$

But the process X admits a continuous version $\widetilde X$ that agrees with X on S with probability one (since S is countable). Replacing X by $\widetilde X$ in the events on the right of (3.23) yields an empty set (because $\widetilde X$ is always continuous). Hence $P(A^{\mathrm{c}}) = 0$ as desired. ☐

Next let us make some remarks on the statement of Theorem 3.4 and its proof. First we note that the observation (3.19) shows that X conforms to:

Definition 3.6 (Stochastic continuity) *A process $\{X_t \colon t \in T\}$ taking values in a metric space $(\mathscr{X}, \varrho)$ is said to be* stochastically continuous *at $t \in T$ if*

$$\forall \epsilon > 0\colon\; \lim_{s\to t} P\big(\varrho(X_t, X_s) > \epsilon\big) = 0 \tag{3.23}$$

Here "$s \to t$" refers to convergence in the index set T.

While (3.7) implies stochastic continuity, it is clearly stronger. We also note that the fact that the exponent on the right of (3.7) exceeds one is essential for the result. This is seen from the example of a *Poisson process* $\{N_t \colon t \in [0,\infty)\}$ defined as

$$N_t := \sup\Big\{m \geqslant 0\colon\; \sum_{k=1}^m Z_k \leqslant t\Big\} \tag{3.24}$$

where $Z_1, Z_2, \dots$ are i.i.d. Exponential(1) random variables. This process takes only integer values and so is discontinuous. It thus fails to obey (3.7), which can also be seen directly by noting that $N_t - N_s$ is Poisson($t-s$) for any $0 \leqslant s < t$ and so, for any $a > 0$,

$$E\big(|N_t - N_s|^a\big) \geqslant P(N_t - N_s \geqslant 1) = 1 - \mathrm{e}^{-(t-s)} \tag{3.25}$$

where the right-hand side decays linearly in $t - s$ as $s \uparrow t$. We will of course use the theorem in the positive direction to prove existence of a continuous version of stochastic processes of interest; particularly, the standard Brownian motion.

The Kolmogorov-Čenstov theorem generalizes quite readily to processes indexed by $t \in \mathbb{R}^d$ — for instance, to a process called the *Brownian sheet*. Here the $1 + b$ exponent in (3.7) needs to be replaced by $d + b$. However, the proof relies too strongly on the "rectangular" geometry of the underlying index set. Other approaches exist, based on careful entropy counting and/or the method of *generic chaining*, that avoid these by "finding" a relevant geometric structure as part of the proof. These turn out to be quite useful in applications but we unfortunately do not have time to discuss them here.

Further reading: Section 2.2B in Karatzas-Shreve