23. "Abstract nonsense" theory of Yamada and Watanabe

In this section we present a theory of Yamada and Watanabe that connects weak and strong solutions and links these with concept of pathwise uniqueness. The theory relies on facts from abstract theory of measure — hence the phrase in the title.

23.1 Uniqueness in law.

The failure of pathwise uniqueness in the Tanaka SDE noted in Proposition 22.7 is actually rather spectacular. To see this, consider a standard Brownian motion X on a probability space $(\Omega, \mathcal{F}, P)$ stared at zero (i.e, $P(X_0 = 0) = 1$) and let $\{\tau_n\}_{n\geqslant 0}$ be a strictly increasing sequence of stopping times for $\{\widetilde{\mathcal{F}}_t^X\}_{t\geqslant 0}$ with $\tau_0 := 0$ and such that $X_{\tau_n} = 0$ for all $n \geqslant 1$. For any sequence $\underline{\alpha} = \{\alpha_n\}_{n\geqslant 1} \in \{-1,1\}^{\mathbb{N}}$, define

$$\forall n \geqslant 1\, \forall t \in [\tau_{n-1}, \tau_n)\colon \quad \widetilde{X}_t^{(\underline{\alpha})} := \alpha_n X_t \tag{23.1}$$

The strong Markov property then shows that also $\{\widetilde{X}_t^{(\underline{\alpha})}\colon t \geqslant 0\}$ is a standard Brownian motion with

$$\int_0^t \operatorname{sgn}(X_s)\mathrm{d}X_s = \int_0^t \operatorname{sgn}(\widetilde{X}_s^{(\underline{\alpha})})\mathrm{d}\widetilde{X}_s^{(\underline{\alpha})} \quad \text{a.s.} \tag{23.2}$$

Writing B_t for the common value of integral (assuming continuous versions), the process $\{B_t\colon t \geqslant 0\}$ is a standard Brownian motion such that

$$\big\{X^{(\underline{\alpha})}\colon \underline{\alpha} \in \{-1,1\}^{\mathbb{N}}\big\} \tag{23.3}$$

are all distinct strong solutions to the Tanaka SDE $\mathrm{d}X_t = \operatorname{sgn}(X_t)\mathrm{d}B_t$ for the standard setting given by the probability space $(\Omega, \mathcal{F}, P)$, filtration $\{\widetilde{\mathcal{F}}_t^X\}_{t\geqslant 0}$, Brownian motion B and the initial value $X_0 = 0$. In particular, since at least one sequence of stopping times as above can be produced — for instance, let $\tau_{n+1} := \inf\{t \geqslant \tau_n + 1\colon X_t = 0\}$ — the Tanaka equation admits a *continuum* of strong solutions in this standard setting.

The above also explains why a strong solution could not be produced on the space with filtration $\{\widetilde{\mathcal{F}}_t^B\}_{t\geqslant 0}$. Indeed, in order to break the degeneracy (in a measurable way) any time a purported solution hits zero we need "additional bits" — represented by $\{\alpha_n\}_{n\geqslant 1}$ above — that are not captured by $\{\widetilde{\mathcal{F}}_t^B\}_{t\geqslant 0}$ in light of (23.2). Increasing the filtration to $\{\widetilde{\mathcal{F}}_t^X\}_{t\geqslant 0}$ solves the existence problem, but only at the cost of having the number of solutions explode.

Notwithstanding the degeneracy, as we showed earlier, any weak solution X of the Tanaka equation is a standard Brownian motion. This suggests an (apparently) weaker notion of uniqueness than what we used so far:

Definition 23.1 (Uniqueness in law) *We say that* uniqueness in law *occurs for an SDE if for any two weak solutions X and $\widetilde{X}$ to this SDE,*

$$X_0 \overset{\text{law}}{=} \widetilde{X}_0 \quad \Rightarrow \quad \{X_t\colon t \geqslant 0\} \overset{\text{law}}{=} \{\widetilde{X}_t\colon t \geqslant 0\} \tag{23.4}$$

Here the equality in law of the stochastic processes X and $\widetilde{X}$ is relative to the Wiener space $(C[0,\infty), \mathcal{B}(C[0,\infty)))$, although equality of finite-dimensional distributions suffices as well.

The two weak solutions may be defined over completely unrelated probability spaces. A somewhat unintuitive feature of the above definition is that the right hand side of (23.4) does not include information about the underlying Brownian motions, but this is the way the concept is used in the literature.

The Tanaka equation is a example of an SDE for which pathwise uniqueness fails but uniqueness in law holds. There are equations in which both concepts fail; for instance

$$\mathrm{d}X_t = 1_{\mathbb{R}\smallsetminus\{0\}}(X_t)\mathrm{d}B_t \tag{23.5}$$

whose solutions are $X_t := B_t$ as well as the process $X_t := 1_{[0,\infty)\smallsetminus[\tau,\tau']}(t)B_t$ for any two stopping times $\tau < \tau'$ with $B_\tau = B_{\tau'} = 0$. While pathwise uniqueness is intuitively stronger than uniqueness in law, their comparison is unclear as inferring uniqueness in law from pathwise uniqueness requires putting unrelated weak solutions on the same probability space. That this can be done was shown by Yamada and Watanabe in 1971 using "abstract non-sense" arguments. The benefit of their method is that it leads to further, and somewhat unexpected, useful consequences. For simplicity of notation, we will only deal with $\mathbb{R}$-valued solutions of SDEs driven by one-dimensional standard Brownian motion.

23.2 Uniformizing map.

We start by defining a map that takes a weak solution X to its natural space, which is the space of continuous $\mathbb{R}$-valued functions $C[0,\infty)$. Since we want to keep track of the standard Brownian motion and the initial value, we will actually use

$$\mathscr{X} := \mathbb{R} \times C[0,\infty) \times C[0,\infty) \tag{23.6}$$

This is endowed with the product Borel σ-algebra

$$\Sigma := \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}\big(C[0,\infty)\big) \otimes \mathcal{B}\big(C[0,\infty)\big) \tag{23.7}$$

For each $t \geqslant 0$ and each triplet $(x_0, \mathrm{b}, \mathrm{x}) \in \mathscr{X}$, where $x_0 \in \mathbb{R}$ while b and x represent elements of $C[0,\infty)$, let

$$\widetilde{X}_t(x_0,\mathrm{b},\mathrm{x}) := \mathrm{x}_t \ \wedge \ \widetilde{B}_t(x_0,\mathrm{b},\mathrm{x}) := \mathrm{b}_t \tag{23.8}$$

denote the canonical coordinate projection maps on $\mathscr{X}$. Setting

$$\forall t \geqslant 0\colon \quad \mathcal{G}_t := \sigma\big(\widetilde{B}_s, \widetilde{X}_s \colon s \leqslant t\big) \tag{23.9}$$

defines a natural filtration $\{\mathcal{G}_t\}_{t\geqslant 0}$ on $(\mathscr{X},\Sigma)$. We will call $(\mathscr{X},\Sigma)$ the *canonical solution space*. We now observe:

Lemma 23.2 (Uniformizing map) *Given a weak solution*

$$\Big((\Omega,\mathcal{F},P),\{\mathcal{F}_t\}_{t\geqslant 0}, B, X\Big) \tag{23.10}$$

set

$$\forall \omega \in \Omega\colon \quad \phi(\omega) := \big(X_0(\omega), B(\omega), X(\omega)\big) \tag{23.11}$$

Then ϕ is a $\mathcal{F}/\Sigma$-measurable map of Ω into $\mathscr{X}$ and $\widetilde{P} := P \circ \phi$ is a probability measure on $(\mathscr{X}, \Sigma)$. Moreover, we have

$$\forall t \geqslant 0\colon \quad \phi^{-1}(\mathcal{G}_t) \subseteq \mathcal{F}_t \tag{23.12}$$

and the following holds: The process $\{\widetilde{B}_t \colon t \geqslant 0\}$ under $\widetilde{P}$ is a standard Brownian motion adapted to $\{\mathcal{G}_t\}_{t\geqslant 0}$ and

$$\Big((\mathscr{X}, \Sigma, \widetilde{P}), \{\mathcal{G}_t\}_{t\geqslant 0}, \widetilde{B}, \widetilde{X} \Big) \tag{23.13}$$

is a weak solution of the SDE.

Proof. That ϕ is a map $\Omega \to \mathscr{X}$ is checked directly from the fact that every path of B and X belong to $C[0,\infty)$. To prove $\mathcal{F}/\Sigma$-measurability of ϕ, it suffices to prove (23.12). Here we note that, for $s \leqslant t$ and $A \in \mathcal{B}(\mathbb{R})$,

$$\phi^{-1}\big(\{\widetilde{B}_s \in A\}\big) = \{B_s \in A\} \in \mathcal{F}_t \tag{23.14}$$

and similarly $\phi^{-1}(\{\widetilde{X}_s \in A\}) \in \mathcal{F}_t$. Now note that $\mathcal{H} := \{C \subseteq \mathscr{X} : \phi^{-1}(C) \in \mathcal{F}_t\}$ is a σ-algebra containing all sets of the form $\{\widetilde{B}_s \in A\}$ and $\{\widetilde{X}_s \in A\}$ for $A \in \mathcal{B}(\mathbb{R})$ and $s \in [0,t]$, and since $\mathcal{G}_t$ is the minimal σ-algebra containing the latter sets, we have $\mathcal{G}_t \subseteq \mathcal{H}$.

By the very definition of $\mathcal{G}_t$, the processes $\widetilde{B}$ and $\widetilde{X}$ are adapted to $\{\mathcal{G}_t\}_{t\geqslant 0}$. Writing $\widetilde{E}$ for expectation with respect to $\widetilde{P}$, the change of variables formula tells us that, for all $0 \leqslant s \leqslant t$, all $A \in \mathcal{G}_s$ and all $h \in C(\mathbb{R})$ bounded,

$$\begin{aligned} \widetilde{E}\big(h(\widetilde{B}_t - \widetilde{B}_s)1_A\big) &= E\big(h(B_t - B_s)1_{\phi^{-1}(A)}\big) \\ &= E\big(h(B_t - B_s)\big)P(\phi^{-1}(A)\big) = E\big(h(B_t - B_s)\big)\widetilde{P}(A) \end{aligned} \tag{23.15}$$

where we used that, since B is a standard Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and, by (23.12), thus also of $\phi^{-1}(A)$. It follows that, under $\widetilde{P}$, the increment $\widetilde{B}_t - \widetilde{B}_s$ is independent of $\mathcal{G}_s$ with $\widetilde{B}_t - \widetilde{B}_s = \mathcal{N}(0, t-s)$ and so $\widetilde{B}$ is a standard Brownian motion with respect to $\{\mathcal{G}_t\}_{t\geqslant 0}$.

It remains to prove that (23.13) is a weak solution. Assume that the equation takes the form $\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t$. As $\widetilde{X}_0 \circ \phi = X_0$, all we need to show that, for each $t \geqslant 0$,

$$\Big(\int_0^t a(s, \widetilde{X}_s)\mathrm{d}s \Big) \circ \phi = \int_0^t a(s, X_s)\mathrm{d}s \quad \text{a.s.} \tag{23.16}$$

and

$$\Big(\int_0^t \sigma(s, \widetilde{X}_s)\mathrm{d}\widetilde{B}_s \Big) \circ \phi = \int_0^t \sigma(s, X_s)\mathrm{d}B_s \quad \text{a.s.} \tag{23.17}$$

For (23.16) the integrals are ordinary Lebesgue integrals for which the identity holds pointwise (no "a.s." is needed). The proof requires checking this for a being an indicator of a set in $\mathcal{B}([0,\infty)) \otimes \mathcal{B}(\mathbb{R})$ and then applying the Monotone Class Theorem.

For (23.16) we first check directly that equality holds for $\sigma := 1_{[a,b]\times A}$ for any $0 < a < b < \infty$ and any $A \in \mathcal{B}(\mathbb{R})$. We then note that, on the (full measure) event where $s \mapsto Y_s := \sigma(s, X_s)$ belongs to $L^2([0,t], \lambda)$, with λ denoting the Lebesgue measure, Y can be approximated (in this L^2) by finite linear combinations of functions of the form

$s \mapsto 1_{[a,b]}(s)1_A(X_s)$. Lemma 11.7 then ensures that this approximation carries over to the Itô integrals thus proving equality (23.17) on a set of full measure. $\square$

23.3 Coupling.

The uniformizing map allows us to push any weak solution (B, X) onto the canonical solution space so that, trivially,

$$(B, X) \text{ under } P \overset{\text{law}}{=} (\widetilde{B}, \widetilde{X}) \text{ under } \widetilde{P} \tag{23.18}$$

Our next task is to "uniformize" two weak solutions at the same time but, to make comparisons meaningful, we need to do it so that the initial value of the solutions and the driving Brownian motion coincide in the target space of the map. The idea here is to use *conditioning* with the help of the following concept:

Definition 23.3 (Regular conditional probability) *Given a probability space $(\mathscr{X}, \Sigma, P)$ and a σ-algebra $\mathcal{G} \subseteq \Sigma$, a* regular conditional probability given $\mathcal{G}$ *is a map*

$$P_{\mathcal{G}} \colon \mathscr{X} \times \Sigma \to [0, 1] \tag{23.19}$$

such that

(1) $\forall x \in \mathscr{X}\colon\ A \mapsto P_{\mathcal{G}}(x, A)$ *is a probability measure on* $(\mathscr{X}, \Sigma)$,
(2) $\forall A \in \Sigma\colon\ \ x \mapsto P_{\mathcal{G}}(x, A)$ *is $\mathcal{G}$-measurable,*
(3) $\forall A \in \Sigma\colon\ P_{\mathcal{G}}(\cdot, A) = E(1_A|\mathcal{G})$ *P-a.s.*

One way to restate (1-3) above is by saying that a regular conditional probability (RCP) is a random measure that expresses (a suitable version of) the conditional expectation $E(\cdot|\mathcal{G})$ as a regular expectation. We leave it to the reader to verify that, just as the conditional expectation, an RCP is unique modulo a modification on a null set.

The existence of RCP is not guaranteed in general but it holds in standard Borel spaces — i.e., measure spaces on completely metrizable separable topological spaces endowed with their Borel sets — of which $(\mathscr{X}, \Sigma)$ is an example. This allows us to proceed as follows: Let

$$\Bigl\{\Bigl((\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)}), \{\mathcal{F}_t^{(i)}\}_{t \geqslant 0}, B^{(i)}, X^{(i)}\Bigr)\Bigr\}_{i=1,2} \tag{23.20}$$

be two solutions to some (possibly different) SDEs such that

$$X_0^{(1)} \overset{\text{law}}{=} X_0^{(2)} \tag{23.21}$$

Let $\widetilde{P}^{(1)}$, resp., $\widetilde{P}^{(2)}$ be the push-forward of the laws $P^{(1)}$, resp., $P^{(2)}$ onto $(\mathscr{X}, \Sigma)$ by the uniformizing map (23.11). Denoting

$$\mathcal{G} := \sigma\Bigl(\bigcup_{t \geqslant 0} \mathcal{G}_t\Bigr) \tag{23.22}$$

for $\mathcal{G}_t$ as in (23.9), let $P_{\mathcal{G}}^{(1)}$, resp., $P_{\mathcal{G}}^{(2)}$ be RCP of $P^{(1)}$, resp., $P^{(2)}$ given $\mathcal{G}$. Since $\mathcal{G}$ does not restrict the third "coordinate" of $\mathscr{X}$, part (2) of Definition 23.3 ensures that the RCPs do not depend on this coordinate. Suppressing that coordinate from notation, we now set

$$\forall A_1, A_2 \in \Sigma\colon \quad Q(A_1 \times A_2) := \int \Bigl(\prod_{i=1,2} \widetilde{P}^{(i)}_{\mathcal{G}}\bigl((x_0,\mathrm{b}), A_i\bigr) \Bigr) \mu(\mathrm{d}x_0) P_W(\mathrm{db}) \tag{23.23}$$

where μ is the common law of $X_0^{(1)}$ and $X_0^{(2)}$ and P_W is the Wiener measure. The motivation for this definition is gleaned from:

Lemma 23.4 (Coupling) *The set function Q extends uniquely to a probability measure on $(\mathscr{X} \times \mathscr{X}, \Sigma \otimes \Sigma)$ and defines a coupling of $\widetilde{P}^{(1)}$ and $\widetilde{P}^{(2)}$ in the sense that*

$$\forall A \in \Sigma\colon \quad Q(A \times \mathscr{X}) = P^{(1)}(A) \;\wedge\; Q(\mathscr{X} \times A) = P^{(2)}(A) \tag{23.24}$$

In addition, writing $(\widetilde{B}^{(1)}, \widetilde{X}^{(1)})$, resp., $(\widetilde{B}^{(1)}, \widetilde{X}^{(1)})$ for the corresponding processes $(\widetilde{B}, \widetilde{X})$ in the first, resp., second term of the product space $\mathscr{X} \times \mathscr{X}$, we have

$$Q\Bigl(\widetilde{X}_0^{(1)} = \widetilde{X}_0^{(2)} \;\wedge\; \forall t \geqslant 0\colon\; \widetilde{B}_t^{(1)} = \widetilde{B}_t^{(2)}\Bigr) = 1 \tag{23.25}$$

Proof. It is easy to check that Q is finitely additive on the semi-algebra $\mathcal{S} := \{A_1 \times A_2\colon A_1, A_2 \in \Sigma)$ on $\mathscr{X} \times \mathscr{X}$, and the argument underlying the proof of Fubini-Tonelli theorem based on the Monotone Convergence Theorem shows that Q is also countably additive on $\mathcal{S}$. The Hahn-Kolmogorov and Dynkin's π/λ-Theorem ensure existence of a unique extension to a probability measure on $\sigma(\mathcal{S}) = \Sigma \times \Sigma$.

The identity (23.24) is then checked readily from (23.23) so it remains to prove (23.25). For this note that Definition 23.3(3) implies

$$\forall i = 1,2\, \forall A \in \mathcal{G}\colon \quad P^{(i)}_{\mathcal{G}}(\,\cdot\,, A) = 1_A \quad P^{(i)}\text{-a.s.} \tag{23.26}$$

Taking $A_1 := \{\widetilde{X}_0^{(1)} > a\}$ and $A_2 := \{\widetilde{X}_0^{(2)} \leqslant a\}$ in (23.23) then shows

$$Q\bigl(\widetilde{X}_0^{(1)} > a \geqslant \widetilde{X}_0^{(2)}\bigr) = \int 1_{\{x_0 > a\}} 1_{\{x_0 \leqslant a\}} \mu(\mathrm{d}x_0) P_W(\mathrm{db}) = 0 \tag{23.27}$$

Taking as union over all $a \in \mathbb{Q}$ shows $\widetilde{X}_0^{(1)} \leqslant \widetilde{X}_0^{(2)}$ Q-a.s. and, by symmetry, $\widetilde{X}_0^{(1)} = \widetilde{X}_0^{(2)}$ Q-a.s. The argument for equality of the processes $\widetilde{B}^{(1)}$ and $\widetilde{B}^{(2)}$ is similar; we just have to prove $\widetilde{B}_t^{(1)} = \widetilde{B}_t^{(2)}$ Q-a.s. for all $t \in \mathbb{Q} \cap [0, \infty)$ and invoke continuity afterwards. $\square$

The previous lemma makes no use of the fact that (23.20) are weak solutions of an SDEs; all we needed was that $B^{(1)}$ and $B^{(2)}$ are standard Brownian motions and that (23.21) was true. We thus also observe:

Lemma 23.5 *Suppose* (23.20) *are weak solutions to (possibly different) SDEs and that* (23.21) *holds. Writing $\mathcal{G}_t^{(1)} := \sigma(\widetilde{X}_s^{(i)}, \widetilde{B}_s^{(i)}\colon s \leqslant t)$, let*

$$\forall t \geqslant 0\colon \quad \mathcal{G}_t := \sigma\bigl(\mathcal{G}_t^{(1)} \cup \mathcal{G}_t^{(2)}\bigr) \tag{23.28}$$

Then

$$\Bigl\{ \Bigl((\mathscr{X} \times \mathscr{X}, \Sigma \otimes \Sigma, Q), \{\mathcal{G}_t\}_{t \geqslant 0}, \widetilde{B}^{(i)}, \widetilde{X}^{(i)} \Bigr) \Bigr\}_{i=1,2} \tag{23.29}$$

are weak solutions of the corresponding SDEs with $\widetilde{X}^{(i)} \overset{\text{law}}{=} X^{(i)}$ *for both* $i = 1, 2$.

Proof. The processes $\widetilde{B}^{(1)}$ and $\widetilde{X}^{(1)}$ are adapted to $\{\mathcal{G}_t^{(1)}\}_{t\geqslant 0}$ and thus also to $\{\mathcal{G}_t\}_{t\geqslant 0}$. Since Q reduces to $\widetilde{P}^{(1)}$ on the first "coordinate" in $\mathscr{X} \times \mathscr{X}$, the fact that, thanks to Lemma 23.4, $\widetilde{X}^{(1)}$ under $\widetilde{P}^{(1)}$ satisfies (2) and (3) in Definition 20.2 carries over to process under Q in the product space. The other process is handled analogously. □

23.4 Main conclusions.

We are now ready to harvest the fruits of our hard labor. Our first result addresses the connection between the two concepts of uniqueness of solutions defined above:

Theorem 23.6 *Pathwise uniqueness implies uniqueness in law.*

Proof. Suppose we have two weak solutions of an SDE. Lemmas 23.4 and 23.5 then allow us to realize these on the same probability space — namely, the product space $\mathscr{X} \times \mathscr{X}$. By pathwise uniqueness, we then have

$$\widetilde{X}^{(1)} = \widetilde{X}^{(2)} \text{ a.s. and thus also } \widetilde{X}^{(1)} \overset{\text{law}}{=} \widetilde{X}^{(2)} \tag{23.30}$$

Since $\widetilde{X}^{(i)} \overset{\text{law}}{=} X^{(i)}$ for both $i = 1, 2$, we conclude that $X^{(1)} \overset{\text{law}}{=} X^{(2)}$, as desired. □

The second result is both useful and conceptually interesting:

Theorem 23.7 *Assume that a weak solution exists and pathwise uniqueness holds for an SDE. Then there exists Borel* $\chi\colon \mathbb{R} \times C[0,\infty) \to C[0,\infty)$ *(with the value of* $\chi(x, f)$ *at* $s \geqslant 0$ *denoted as* $\chi_s(x, f)$*) such that for any probability space* $(\Omega, \mathcal{F}, P)$ *supporting a standard Brownian motion* B *and a Brownian filtration* $\{\mathcal{F}_t\}_{t\geqslant 0}$*, we have*

$$\forall t \geqslant 0\colon \quad \sigma(\chi_s\colon s \leqslant t) \subseteq \sigma\Big(\{x_0 \in A_1\}, \{b_s \in A_2\}\colon\ s \leqslant t\ \wedge\ A_1, A_2 \in \mathcal{B}(\mathbb{R})\Big) \tag{23.31}$$

and also that

$$X := \chi(X_0, B) \tag{23.32}$$

is a strong solution to the SDE for the standard setting $((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}_{t\geqslant 0}, B, X_0)$.

Proof. Consider a weak solution X on the probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and Brownian motion denoted by B. Pick a version $P_{\mathcal{G}}$ of regular conditional probability given $\mathcal{G}$, for $\mathcal{G}$ as in (23.22) where $\mathcal{G}_t$ is as in (23.9). For each $a \in \mathbb{R}$, $t \geqslant 0$, $x_0 \in \mathbb{R}$ and $\mathrm{b} \in C[0,\infty)$, abbreviate

$$\varphi_{t,a}(x_0, \mathrm{b}) := \widetilde{P}_{\mathcal{G}}\big((x_0, \mathrm{b}), \{\widetilde{X}_t \leqslant a\}\big) \tag{23.33}$$

where $\widetilde{P}$, resp., $\widetilde{X}$ is the probability, resp., the solution on the canonical solution space. We will need two observations about this object:

Lemma 23.8 *For all* $a \in \mathbb{R}$,

$$\varphi_{t,a}(X_0, B) \in \{0, 1\} \quad P\text{-a.s} \tag{23.34}$$

Proof. Use Lemma 23.4 to uniformize and couple the solution with itself on $\mathscr{X} \times \mathscr{X}$. Writing $\widetilde{X}^{(1)}$, resp, $\widetilde{X}^{(2)}$ for the two solutions, pathwise uniqueness implies

$$\widetilde{X}^{(1)} = \widetilde{X}^{(2)} \quad Q\text{-a.s.} \tag{23.35}$$

and so a calculation similar to (23.27) shows

$$\begin{aligned} 0 &= Q\big(\widetilde{X}_t^{(1)} < a \leqslant \widetilde{X}_t^{(2)}\big) \\ &= \int \widetilde{P}_{\mathcal{G}}\big((x_0,\mathrm{b}),\{\widetilde{X}_t \leqslant a\}\big)\Big[1-\widetilde{P}_{\mathcal{G}}\big((x_0,\mathrm{b}),\{X_t \leqslant a\}\big)\Big]\mu(\mathrm{d}x_0)P_W(\mathrm{db}) \end{aligned} \tag{23.36}$$

Since $u(1-u) = 0$ implies $u = \pm 1$, the claim follows. □

Lemma 23.9 *For all $t \geqslant 0$ and all $a > 0$,*

$$P\Big(\{X_t > a\} \cap \big\{\varphi_{t,a}(X_0, B) = 1\big\}\Big) = 0 \tag{23.37}$$

and

$$P\Big(\{X_t \leqslant a\} \cap \big\{\varphi_{t,a}(X_0, B) = 0\big\}\Big) = 0 \tag{23.38}$$

Proof. Since $\varphi_{t,a}(X_0, B)$ is $\mathcal{G}$-measurable and $\widetilde{P}_{\mathcal{G}}(\cdot, A)$ is a version of $\widetilde{E}(1_A|\mathcal{G})$, where $\widetilde{E}$ is the expectation with respect to the probability $\widetilde{P}$ pushed onto the canonical solution space by the uniformizing map. For the first probability we then get

$$\begin{aligned} P\big(\{X_t > a\} \cap \{\varphi_{t,a}(X_0, B) = 1\big) &= \widetilde{P}\big(\{\widetilde{X}_t > a\} \cap \{\varphi_{t,a}(\widetilde{X}_0, \widetilde{B}) = 1\big) \\ &= \widetilde{E}\Big(\widetilde{P}_{\mathcal{G}}\big(\,\cdot\,\big|\,\{\widetilde{X}_t > a\}\big)1_{\{\varphi_{t,a}(\widetilde{X}_0,\widetilde{B})=1\}}\Big) \\ &= E\Big(\big[1-\varphi_{t,a}(X_0,B)\big]1_{\{\varphi_{t,a}(X_0,B)=1\}}\Big) = 0 \end{aligned} \tag{23.39}$$

The second probability is handled similarly. □

The above lemmas imply that, denoting

$$\psi_t(x_0, \mathrm{b}) := \inf\big\{a \in \mathbb{Q}\colon\ \varphi_{t,a}(x_0, \mathrm{b}) = 1\big\} \tag{23.40}$$

for each $t \geqslant 0$ we have

$$X_t = \psi_t(X_0, B) \quad P\text{-a.s.} \tag{23.41}$$

where the restriction to a rational in (23.40) is needed to overcome confluence of uncountably many null sets and/or lack of measurability. In particular, since $t \mapsto X_t$ is continuous by assumption, the infimum on the right is locally uniformly continuous on the rationals meaning that the total variation in t over $[0, T]$,

$$V_T^{(1)}\big(\psi(x_0, \mathrm{b}), \epsilon\big) := \sup_{\substack{t,s \in \mathbb{Q} \cap [0,T] \\ |t-s| < \epsilon}} \big|\psi_t(x_0, \mathrm{b}) - \psi_s(x_0, \mathrm{b})\big| \tag{23.42}$$

obeys

$$\lim_{\epsilon \downarrow 0} V_T^{(1)}\big(\psi(X_0, B), \epsilon\big) = 0 \quad P\text{-a.s.} \tag{23.43}$$

for all $T \geqslant 1$.

Moving, finally, towards the construction of the desired map, consider the sets

$$\widetilde{N}_1 := \bigcap_{a\in\mathbb{Q}} \Big\{ (x_0, \mathrm{b}, \mathrm{x}) \in \mathscr{X} \colon\ \varphi_{\mathcal{G}}\big((x_0,\mathrm{b}), \{X_t \leqslant a\}\big) \notin \{0,1\} \Big\} \tag{23.44}$$

and

$$\widetilde{N}_2 := \bigcap_{n\geqslant 1} \Big\{ (x_0, \mathrm{b}, \mathrm{x}) \in \mathscr{X} \colon\ \lim_{\epsilon\downarrow 0} V_n^{(1)}\big(\psi(x_0,\mathrm{b}), \epsilon\big) > 0 \Big\} \tag{23.45}$$

and, writing ϕ for the uniformizing map, set $N_1 := \phi^{-1}(\widetilde{N}_1)$ and $N_2 := \phi^{-1}(\widetilde{N}_2)$. The above observations guarantee

$$N_1, N_2 \in \mathcal{G} \ \wedge\ P(N_1 \cup N_2) = 0 \tag{23.46}$$

Setting, for any $t \geqslant 0$, $x_0 \in \mathbb{R}$ and $\mathrm{b} \in C[0,\infty)$,

$$\chi_t(x_0,\mathrm{b}) := \begin{cases} \psi_t(x_0,\mathrm{b}) & \text{on } \mathscr{X} \smallsetminus (\widetilde{N}_1 \cup \widetilde{N}_2) \\ x_0 & \text{on } \widetilde{N}_1 \cup \widetilde{N}_2 \end{cases} \tag{23.47}$$

we get a continuous map $t \mapsto \chi_t(x_0,\mathrm{b})$ such that, for each $t \geqslant 0$,

$$(x_0,\mathrm{b}) \mapsto \chi_t(x_0,\mathrm{b}) \ \text{ is } \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(C[0,t])/\mathcal{B}(\mathbb{R})\text{-measurable} \tag{23.48}$$

where $\mathcal{B}(C[0,t])$ is the sub-σ-algebra of $\mathcal{B}(C[0,\infty))$ on the right-hand side of (23.31). Writing $\chi(x_0,\mathrm{b})$ for the function $t \mapsto \chi_t(x_0,\mathrm{b})$, since $\mathcal{B}(C[0,\infty))$ is generated by sets of the form $\{f \in C[0,\infty) \colon f(t) \in A\}$ for $A \in \mathcal{B}(\mathbb{R})$, it follows that

$$(x_0,\mathrm{b}) \mapsto \chi(x_0,\mathrm{b}) \ \text{ is } \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(C[0,\infty))/\mathcal{B}(C[0,\infty))\text{-measurable} \tag{23.49}$$

and that (23.31) holds

In light of (23.41) and (23.46), $X_t = \chi_t(X_0, B)$ P-a.s. holds for each $t \in \mathbb{Q}$. Since both X and $\chi(X_0,B)$ are continuous, this implies $X = \chi(X_0,B)$ a.s. We now check that, for any standard setting with initial value X_0 and B a standard Brownian motion, $t \mapsto \chi_t(X_0,B)$ is a strong solution. $\square$

We have thus shown that, under the condition of pathwise uniqueness, if there is at least one weak solution, that and every other solution can be described by a *solution map*.

Further reading: Karatzas-Shreve, Section 5.3CD