22. Tanaka equation and weak solutions

In the previous lecture we discussed local solutions — or, more precisely, solutions defined up to, and sometimes including, a stopping time — and established their existence and uniqueness under the local Lipschitz property of the coefficients. Here we will analyze a simple example where these conclusion fail, naturally leading us to the concept of a weak solution.

22.1 Tanaka's example.

Let us first formalize the concept of uniqueness established under the Lipschitz conditions on the coefficients in Theorem 21.4:

Definition 22.1 (Pathwise uniqueness) *We say that a strong solution up to stopping time is* unique in a pathwise sense *(for a given standard setting) if for any strong solutions* $\{X_t \colon t \in [0,T]\}$ *and* $\{\widetilde{X}_t \colon t \in [0,\widetilde{T}]\}$ *up to finite (stopping) times* T *and* $\widetilde{T}$*, respectively,*

$$P(X_0 = \widetilde{X}_0) = 1 \tag{22.1}$$

implies

$$P\Big(\forall t \in [0, T \wedge \widetilde{T}] \colon X_t = \widetilde{X}_t\Big) = 1 \tag{22.2}$$

We will also say that a strong solution is globally pathwise unique *if the above applies for stopping times* $T = \widetilde{T} := n$ *for all* $n \geqslant 1$.

As for the strong solution, the concept of pathwise uniqueness is attached to a given *instance* of the standard setting for the SDE of interest, although some treatments (such as Karatzas-Shreve) allow the filtration for $\widetilde{X}$ to be different from that for X.

While Theorem 21.4 works under the local Lipschitz assumption on the coefficients, a natural question is whether strong solutions exist and are pathwise unique for other coefficients as well. The answer is of course negative as this is not even true for ordinary differential equations that are naturally included, by setting $\sigma = 0$, in our class of SDEs. But how about equations of the form

$$\mathrm{d}X_t = \sigma(t, X_t)\mathrm{d}B_t \tag{22.3}$$

which take us as far as an SDE can be from an ODE? As it turns out, this is not the case either but the corresponding analysis is more subtle.

As already observed before, with each stochastic process $\{Y_t \colon t \geqslant 0\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ one can associate a *natural filtration* $\{\mathcal{F}_T^Y\}_{t \geqslant 0}$, where

$$\forall t \geqslant 0\colon \quad \mathcal{F}_t^Y := \sigma(Y_s \colon s \leqslant t) \tag{22.4}$$

be the least σ-algebra that makes all random variables in $\{Y_s \colon s \leqslant t\}$ measurable. For reasons that have been discussed earlier, this is usually not enough for the purposes of stochastic analysis, so writing

$$\mathcal{N} := \big\{A \in \mathcal{F} \colon P(A) = 0\big\} \tag{22.5}$$

for the class of P-null sets, we enlarge the natural filtration to the *augmented filtration* $\{\widetilde{\mathcal{F}}_t^Y\}_{t\geqslant 0}$ by setting

$$\forall t \geqslant 0\colon \quad \widetilde{\mathcal{F}}_t^Y := \sigma\big(\mathcal{N} \cup \mathcal{F}_t^Y\big) \tag{22.6}$$

A key point for us is that, if B is a standard Brownian motion, then $\{\widetilde{\mathcal{F}}_t^B\}_{t\geqslant 0}$ remains to be a Brownian filtration. With these in hand, we now claim:

Proposition 22.2 (Tanaka SDE) *Consider a standard setting with a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $\{B_t\colon t \geqslant 0\}$, the augmented Brownian filtration $\{\widetilde{\mathcal{F}}_t^B\}$ and initial value $X_0 := 0$. Then the Tanaka SDE*

$$\mathrm{d}X_t = \mathrm{sgn}(X_t)\mathrm{d}B_t \tag{22.7}$$

where

$$\mathrm{sgn}(x) := \begin{cases} 1 & \text{if } x \geqslant 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{22.8}$$

does not admit a strong solution.

That this should be the case is witnessed by an attempt to construct a solution via gluing together pieces of B. Notice that the SDE dictates that, the process X moves exactly as an increment of path B while non-negative and as increment of path $-B$ when it is negative. This shows that $X_t := X_0 + B_t$ is a definitely a solution up to the stopping time $\tau_0 := \inf\{s \geqslant 0\colon X_0 + B_s = 0\}$ and continues to be even if the "next" excursion of $t \mapsto B_t - B_\tau$ from zero lies "above" zero. The problem is when that excursion lies "below" zero because then, if X were to remain non-negative then the SDE asks it to "follow" the increment of B which would make it negative, while if X were to become negative immediately after $t = \tau$, then the SDE asks it to "follow" the negative of the increment of B, which would make it positive.

A problem with this reasoning is that the notion of "next" excursion of the Brownian motion is not meaningful; indeed, the Brownian motion will see (countably) infinitely many such excursion in any interval of positive length after time τ_0. We will thus have to proceed using methods stochastic calculus instead.

22.2 Tanaka's formula and proof of Proposition 22.2.

The proof of Proposition 22.2 is based on a lemma that is of independent interest for a later discussion of *Brownian local time*:

Lemma 22.3 (Tanaka formula) *Let B denote a standard Brownian motion. Then*

$$\forall t \geqslant 0\colon \quad \frac{1}{2\epsilon}\int_0^t 1_{\{|B_s|<\epsilon\}}\mathrm{d}s \;\underset{\epsilon\downarrow 0}{\overset{P}{\longrightarrow}}\; |B_t| - |B_0| - \int_0^t \mathrm{sgn}(B_s)\mathrm{d}B_s \tag{22.9}$$

If $\{\epsilon_n\}_{n\geqslant 1}$ are positive with $\sum_{n\geqslant 1}\epsilon_n < \infty$, the limit takes place a.s. along this sequence.

Proof. The last two terms on the right-hand side coincide with the first two terms of the Itô formula for $f(B_t)$, where $f(z) := |z|$, because $f'(z) = \mathrm{sgn}(z)$ for $z \neq 0$. However, this f is not in $C^2(\mathbb{R})$ and so the Itô formula cannot be used. That being said, it turns out that it can be used for f replaced by

$$f_\epsilon(x) := \begin{cases} \frac{\epsilon}{2} + \frac{x^2}{2\epsilon} & \text{if } |x| \leqslant \epsilon \\ |x| & \text{if } |x| > \epsilon \end{cases} \tag{22.10}$$

Indeed, f_ϵ' exists on $\mathbb{R}$ with

$$\forall x \in \mathbb{R}\colon \quad f_\epsilon'(x) := \begin{cases} \frac{x}{\epsilon} & \text{if } |x| \leqslant \epsilon \\ \operatorname{sgn}(x) & \text{if } |x| > \epsilon \end{cases} \tag{22.11}$$

and even f_ϵ'' exists away from $x = \pm\epsilon$ with

$$\forall x \neq \pm\epsilon\colon \quad f_\epsilon(x) = \frac{1}{\epsilon} 1_{(-\epsilon,\epsilon)}(x) \tag{22.12}$$

This puts f_ϵ within the scope of:

Lemma 22.4 *Let $B = \{B_t\colon t \geqslant 0\}$ be a standard Brownian motion. For all $f \in C^1$ with f' absolutely continuous,*

$$\forall t \geqslant 0\colon \quad f(B_t) = f(B_0) + \int_0^t f'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s \quad \text{a.s.} \tag{22.13}$$

where f'' is (any representative) of the Lebesgue derivative of f'.

Postponing the proof of this lemma until a later section, we thus have

$$f_\epsilon(B_t) = f_\epsilon(B_0) + \int_0^t f_\epsilon'(B_s)\mathrm{d}B_s + \frac{1}{2\epsilon}\int_0^t 1_{\{|B_s|<\epsilon\}}\mathrm{d}s \tag{22.14}$$

We now want to take the limit as $\epsilon \downarrow 0$. For this observe that

$$\forall x \in \mathbb{R}\colon \quad |x| \leqslant f_\epsilon(x) \leqslant |x| \vee \epsilon \tag{22.15}$$

and so

$$\forall x \in \mathbb{R}\colon \quad \lim_{\epsilon\downarrow 0} f_\epsilon(x) = |x| \tag{22.16}$$

In particular $f_\epsilon(B_t) \to |B_t|$ and $f_\epsilon(B_0) \to |B_0|$ pointwise and so, in order to prove the claim, we thus need to show that

$$\int_0^t f_\epsilon'(B_s)\mathrm{d}B_s \ \underset{\epsilon\downarrow 0}{\overset{P}{\longrightarrow}} \ \int_0^t \operatorname{sgn}(B_s)\mathrm{d}B_s \tag{22.17}$$

For this we note that, by Itô isometry and the fact that

$$\forall x \in \mathbb{R}\colon \quad \bigl|f_\epsilon(x) - \operatorname{sgn}(x)\bigr| \leqslant 1_{(-\epsilon,\epsilon)}(x) \tag{22.18}$$

we have

$$\begin{aligned} E\biggl(\Bigl|\int_0^t f_\epsilon'(B_s)\mathrm{d}B_s - \int_0^t \operatorname{sgn}(B_s)\mathrm{d}B_s\Bigr|^2\biggr) &= E\int_0^t \bigl|f_\epsilon(B_s) - \operatorname{sgn}(B_s)\bigr|^2\mathrm{d}s \\ &\leqslant E\int_0^t 1_{\{|B_s|<\epsilon\}}\mathrm{d}s = \int_0^t P\bigl(|B_s|<\epsilon\bigr)\mathrm{d}s \leqslant 2\epsilon\int_0^t \frac{1}{\sqrt{s}}\mathrm{d}s \end{aligned} \tag{22.19}$$

where we used Fubini-Tonelli to exchange the expectation with the integral over s and then noted that the probability density of B_s is bounded by $\frac{1}{\sqrt{s}}$. Since the integral is convergent for all $t \geqslant 0$, the right-hand side tends to zero as $\epsilon \downarrow 0$ thus proving that the convergence in (22.17) takes place even in L^2. The second clause then follows by combining (22.19) with Chebyshev's inequality and the Borel-Cantelli lemma. □

We are now ready to give:

Proof of Proposition 22.2. Suppose, aiming at a contradiction, that the Tanaka equation admits a strong solution $\{X_t : t \geqslant 0\}$ with some initial value X_0. Then

$$\forall t \geqslant 0\colon \quad X_t = X_0 + \int_0^t \operatorname{sgn}(X_s)\mathrm{d}B_s \tag{22.20}$$

This implies

$$\forall t \geqslant 0\colon \quad \langle X\rangle_t = \int_0^t \operatorname{sgn}(X_s)^2\mathrm{d}s = \int_0^t \mathrm{d}s = t \tag{22.21}$$

Lévy's characterization of standard Brownian motion (see Theorem 14.6) then tells us that X is a standard Brownian motion.

Itô calculus for semimartingales in turn tells us that

$$\int_0^t \operatorname{sgn}(X_s)\mathrm{d}X_s = \int_0^t \operatorname{sgn}(X_s)^2\mathrm{d}B_s = \int_0^t \mathrm{d}B_s = B_t \quad \text{a.s.} \tag{22.22}$$

while the Tanaka formula from Lemma 22.3 gives

$$\int_0^t \operatorname{sgn}(X_s)\mathrm{d}X_s = |X_t| - |X_0| - \lim_{n\to\infty}\frac{1}{2\epsilon_n}\int_0^t 1_{\{|X_s|<\epsilon_n\}}\mathrm{d}s \quad \text{a.s.} \tag{22.23}$$

where the choice $\epsilon_n := 1/n^2$ ensures that the convergence is a.s.

The upshot of these estimates is that B_t is expressed as a function of $\{|X_s| : s \leqslant t\}$ almost surely. This shows that B is adapted to the filtration $\{\widetilde{\mathcal{F}}_t^{|X|}\}_{t\geqslant 0}$ meaning that

$$\forall t \geqslant 0\colon \quad \widetilde{\mathcal{F}}_t^{B} \subseteq \widetilde{\mathcal{F}}_t^{|X|} \tag{22.24}$$

But the fact that X a strong solution for the stated setting means that

$$\forall t \geqslant 0\colon \quad \widetilde{\mathcal{F}}_t^{X} \subseteq \widetilde{\mathcal{F}}_t^{B} \tag{22.25}$$

whereby we conclude

$$\forall t \geqslant 0\colon \quad \widetilde{\mathcal{F}}_t^{X} \subseteq \widetilde{\mathcal{F}}_t^{|X|} \tag{22.26}$$

To see that this is absurd, we note:

Lemma 22.5 *For X a standard Brownian motion with $X_0 = 0$,*

$$\forall t > 0\colon \{X_t > 0\} \notin \widetilde{\mathcal{F}}_t^{|X|} \tag{22.27}$$

Indeed, as $\{X_t > 0\} \in \widetilde{\mathcal{F}}_t^{X}$, Lemma 22.5 shows that (22.26) is false. We conclude that no strong solution to the Tanaka equation with $X_0 = 0$ exists as claimed. □

We owe the reader:

Proof of Lemma 22.5. While this is intuitive, the proof requires analysis. Fix $t > 0$ and let

$$\mathcal{G}_t := \big\{A \in \widetilde{\mathcal{F}} \colon E(X_t 1_A) = 0\big\} \tag{22.28}$$

Using that $X_t \in L^1$ and $E(X_t) = 0$, the additivity of the integral implies that $\mathcal{G}_t$ is a σ-algebra. We now observe that, for any $B \in \mathcal{B}(\mathbb{R})$ and any $s \leqslant t$, the reflection symmetry of Brownian paths implies

$$E\big(X_1 1_{\{|X_s| \in B\}}\big) = 0 \tag{22.29}$$

thus showing that $\{|X_s| \in B\} \in \mathcal{G}_t$. It follows that $\mathcal{F}_t^{|X|} \subseteq \mathcal{G}_t$ which, in light of (22.6) and $\mathcal{N} \subseteq \mathcal{G}_t$, proves $\widetilde{\mathcal{F}}_t^{|X|} \subseteq \mathcal{G}_t$. To get the claim, notice that

$$E\big(X_t 1_{\{X_t > 0\}}\big) > 0 \tag{22.30}$$

holds $t > 0$. Hence we get $\{X_t > 0\} \notin \mathcal{G}_t$ and so $\{X_t > 0\} \notin \widetilde{\mathcal{F}}_t^{|X|}$. $\square$

Notice that the assumption that $\mathrm{sgn}(x)^2 = 1$ was used crucially in the proof. That this matters even for the statement, define

$$\mathrm{sign}(x) := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{22.31}$$

Then

$$X_t := X_0 + \mathrm{sign}(X_0) B_{t \wedge \tau_0} \tag{22.32}$$

where

$$\tau_0 := \inf\{s \geqslant 0 \colon X_0 + \mathrm{sign}(X_0) B_{t \wedge \tau_0} = 0\} \tag{22.33}$$

obeys $\mathrm{d}X_t = 1_{\{\tau_0 > t\}} \mathrm{sign}(X_0) \mathrm{d}B_t$ which in light of the fact that $\mathrm{sign}(X_t) = \mathrm{sign}(X_0)$ on the event $\{\tau_0 > t\}$ shows that it is a strong solution to the SDE $\mathrm{d}X_t = \mathrm{sign}(X_t)\mathrm{d}B_t$.

22.3 Weak solutions.

While the absence of a strong solution to Tanaka's equation may be disappointing, note that a solution can still be produced. Indeed, let X be a standard Brownian motion started from X_0 and, inspired by (22.22), set

$$B_t := \int_0^t \mathrm{sgn}(X_t) \mathrm{d}X_t \tag{22.34}$$

The Itô calculus then tells us that

$$\int_0^t \mathrm{sgn}(X_t)\mathrm{d}B_t = \int_0^t \mathrm{sgn}(X_t)^2 \mathrm{d}X_t = \int_0^t \mathrm{d}X_t = X_t - X_0 \tag{22.35}$$

thus showing that the pair (X, B) satisfies the Tanaka SDE and is thus a strong solution on the probability space carrying the standard Brownian motion X. In fact, the only important difference compared to the setting of Proposition 22.2 is the filtration, which needs to be at least as large as $\{\widetilde{\mathcal{F}}_t^X\}_{t \geqslant 0}$ here. So, all that was needed to get a strong solution was to enlarge the filtration!

The solution we just presented falls under a general framework of:

Definition 22.6 (Weak solution) *A* weak solution *of the SDE* (20.7) *is a four-tuple*

$$\Big((\Omega,\mathcal{F},P),\{\mathcal{F}_t\}_{t\geqslant 0},B,X\Big) \tag{22.36}$$

where $(\Omega,\mathcal{F},P)$ *is a probability space,* $B=\{B_t\colon t\geqslant 0\}$ *is a standard Brownian motion,* $\{\mathcal{F}_t\}_{t\geqslant 0}$ *is a Brownian filtration and* X *is a strong solution to the SDE for the standard setting determined by* $(\Omega,\mathcal{F},P)$*,* $\{\mathcal{F}_t\}_{t\geqslant 0}$ *and* B*.*

An important difference compared to the strong solution is that the standard setting is now part of the solution. Note that the concept of pathwise uniqueness still makes sense here except that, in order to claim its validity for the weak solution, we have to examine *all* possible standard settings for which a strong solution exists. Also here the Tanaka SDE serves as an interesting counterexample:

Proposition 22.7 *Suppose* (X,B) *is a pair of Brownian motions on a probability space such that, under a Brownian (w.r.t. B) filtration* $\{\mathcal{F}_t\}_{t\geqslant 0}$*,* X *is a solution to the Tanaka SDE. Let*

$$\tau := \inf\{t\geqslant 1\colon X_t=0\} \tag{22.37}$$

and for each $t\geqslant 0$ *define*

$$\widetilde{X}_t := \begin{cases} X_t & \text{if } t\leqslant\tau\\ -X_t & \text{if } t>\tau\end{cases} \tag{22.38}$$

Then $\widetilde{X}$ *is also a solution to the Tanaka SDE under the same filtration and Brownian motion and, since* $\tau<\infty$ *a.s.,* $\widetilde{X}\neq X$*. In short, pathwise uniqueness does not hold for the Tanaka equation.*

Proof. The process $\widetilde{X}$ is clearly continuous, so we have to check that it is adapted. This follows from

$$\{\widetilde{X}_t\in A\}=\Big(\{X_t\in A\}\cap\{\tau\leqslant t\}\Big)\cup\Big(\{X_t\in -A\}\cap\{\tau\leqslant t\}\Big) \tag{22.39}$$

and the fact that $\{X_t\in\pm A\}\in\mathcal{F}_t$ when $A\in\mathcal{B}(\mathbb{R}^d)$ by the fact that X is adapted, and that $\{\tau\leqslant t\}\in\mathcal{F}_t$ by the fact that τ is a stopping time.

Since $\widetilde{X}$ is a Brownian motion, we have $\operatorname{sgn}(\widetilde{X}_s)=[2\cdot 1_{\{\tau>s\}}-1]\operatorname{sgn}(X_s)$ for Lebesgue a.e. $s\geqslant 0$. Hence we get

$$\begin{aligned}\int_0^t \operatorname{sgn}(\widetilde{X}_s)\mathrm{d}B_s &= \int_0^t \big[2\cdot 1_{\{\tau>s\}}-1\big]\operatorname{sgn}(X_s)\mathrm{d}B_s\\ &= 2\int_0^{\tau\wedge t}\operatorname{sgn}(X_s)\mathrm{d}B_s-\int_0^t\operatorname{sgn}(X_s)\mathrm{d}B_s\\ &= 2(X_{\tau\wedge t}-X_0)-(X_t-X_0)=\widetilde{X}_t-\widetilde{X}_0,\end{aligned} \tag{22.40}$$

where in the second to last equality we used that X is a solution to the Tanaka SDE for the underlying Brownian motion given by B. This shows that $\widetilde{X}$ is a solution to the Tanaka SDE over the same standard setting as X. Pathwise uniqueness fails in this example. □

Further reading: Karatzas-Shreve, Section 5.3