## 21. Uniqueness and localily

Having furnished criteria for existence of a strong solution, we now address uniqueness. Along with that we also prove locality, which means that the solution depends only on the part of the coefficients that it has"seen" so far.

### 21.1 Main statement.

An unfortunate feature of Theorem 20.3 is that its assumptions are often too restrictive. Indeed, the setting does not even include the Bessel SDE, which is the only SDE we have seriously considered so far. So before even beginning to discuss uniqueness of strong solutions, we need to generalize the concept itself:

**Definition 21.1** (Strong solution up to a stopping time) *Assume a standard setting for the SDE* (20.7) *with notations as in Definition 20.1. Given a finite stopping time $T$ for the underlying filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, saying that $\{X_t\colon t\in[0,T]\}$ is a* strong solution *to* (20.7) up to time $T$ *means*

1. $\{X_{T\wedge t}\colon t\geqslant 0\}$ *is continuous and adapted to* $\{\mathcal{F}_t\}_{t\geqslant 0}$,
2. *for all* $t\geqslant 0$,
$$\int_0^t 1_{\{T>s\}}\big|a(s,X_s)\big|\mathrm{d}s<\infty \;\wedge\; \int_0^t 1_{\{T>s\}}\big|\sigma(s,X_s)\big|^2\mathrm{d}s<\infty \quad \text{a.s.} \tag{21.1}$$
3. *for all* $t\geqslant 0$,
$$X_{T\wedge t}=X_0+\int_0^t 1_{\{T>s\}}\,a(s,X_s)\mathrm{d}s+\int_0^t 1_{\{T>s\}}\sigma(s,X_s)\cdot\mathrm{d}B_s \quad \text{a.s.} \tag{21.2}$$

   *where $X_0$ on the right denotes the initial value prescribed in the standard setting.*

*In addition, given a $\mathbb{R}_+\cup\{+\infty\}$-valued random variable $T$, we say that $\{X\colon t\in[0,T)\}$ is a strong solution up to a time $T$ if there exists a sequence $\{T_n\}_{n\geqslant 1}$ of finite stopping times such that*
$$\forall n\geqslant 1\colon\quad T_n\leqslant T_{n+1} \tag{21.3}$$
*and*
$$T=\lim_{n\to\infty}T_n \tag{21.4}$$
*and such that, for each $n\geqslant 1$, $\{X_t\colon t\in[0,T_n]\}$ is a strong solution up to time $T_n$.*

In the first part require $T$ to be finite in order to allow for $X_T$ to be defined. This is mended in the second part. Note that $\{T>t\}=\bigcup_{n\geqslant 0}\{T_n>t\}$ so $T$ is a stopping time in this case as well. Taking $T:=\infty$ and $T_n:=n$ we easily check that the above subsumes the concept of a strong solution from Definition 20.1. We can also take $T$ deterministic, which then defines give the concept of "a strong solution up to a fixed time."

**Theorem 21.2** (Uniqueness and locality) *Assume a standard setting with same filtered probability space and Brownian motion, but two pairs of coefficients $(a,\sigma)$ and $(\tilde{a},\tilde{\sigma})$ satisfying (20.18–20.19). Let $D\subseteq\mathbb{R}^d$ be non-empty open with*
$$\forall t\geqslant 0\,\forall x\in D\colon\quad a(t,x)=\tilde{a}(t,x)\;\wedge\;\sigma(t,x)=\tilde{\sigma}(t,x) \tag{21.5}$$

*If $\{X_t\colon t\in[0,T]\}$ is a strong solution up to (stopping) time $T$ to SDE* (20.7) *with coefficients $(a,\sigma)$ and $\{\widetilde{X}_t\colon t\in[0,\widetilde{T}]\}$ is a strong solution up to (stopping) time $\widetilde{T}$ to SDE* (20.7) *with coefficients $(\tilde a,\tilde\sigma)$ but possibly different initial value, then for*

$$\tau := T\wedge\widetilde{T}\wedge\inf\bigl\{t\geqslant 0\colon X_t\notin D\ \vee\ \widetilde{X}_t\notin D\bigr\} \tag{21.6}$$

*we have*

$$\forall t\geqslant 0\,\exists C(t)<\infty\colon\quad E\Bigl(\sup_{s\leqslant t\wedge\tau}|X_s-\widetilde{X}_s|^2\Bigr)\leqslant C(t)E\bigl(|X_0-\widetilde{X}_0|^2\bigr) \tag{21.7}$$

*In particular,*

$$P(X_0=\widetilde{X}_0)=1\ \Rightarrow\ P\Bigl(\forall t\geqslant 0\colon\ X_{\tau\wedge t}=\widetilde{X}_{\tau\wedge t}\Bigr)=1 \tag{21.8}$$

*Proof.* The argument is actually very similar to that used in the construction of a strong solution. Indeed, suppose $X$ and $\widetilde{X}$ are two strong solutions and write $\tau_r := \tau\wedge B(0,r)$ for any $r>0$ where $B(0,r):=\{x\in\mathbb{R}^d\colon |x|<r\}$ Then

$$\forall t\geqslant 0\colon\quad X_{\tau_r\wedge t}-\widetilde{X}_{\tau_r\wedge t}=X_0-\widetilde{X}_0+A_t+M_t \tag{21.9}$$

where, this time,

$$A_t := \int_0^t 1_{\{\tau_r>s\}}\bigl[a(s,X_s)-a(s,\widetilde{X}_s)\bigr]\mathrm{d}s \tag{21.10}$$

and

$$M_t := \int_0^t 1_{\{\tau_r>s\}}\bigl[\sigma(s,X_s)-\sigma(s,\widetilde{X}_s)\bigr]\cdot\mathrm{d}B_s \tag{21.11}$$

where we used that, on $\{\tau_r>s\}$, we have $X_s,\widetilde{X}_s\in D$ and so $\tilde a(s,X_s)=a(s,\widetilde{X}_s)$ and $\tilde\sigma(s,X_s)=\sigma(s,\widetilde{X}_s)$. Using the inequality $(a+b+c)^2\leqslant 3a^2+3b^2+3c^2$ we get

$$\sup_{s\leqslant t\wedge\tau_r}|X_s-\widetilde{X}_s|^2\leqslant 3|X_0-\widetilde{X}_0|^2+3\sup_{s\leqslant t}A_s^2+3\sup_{s\leqslant t}M_s^2 \tag{21.12}$$

Denoting

$$g_r(t) := E\Bigl(\sup_{s\leqslant t\wedge\tau_r}|X_s-\widetilde{X}_s|^2\Bigr) \tag{21.13}$$

the estimates (20.36–20.37) then show

$$g_r(t)\leqslant 3E\bigl(|X_0-\widetilde{X}_0|^2\bigr)+3K^2(t+4)\int_0^t g_r(s)\mathrm{d}s \tag{21.14}$$

We now invoke:

**Lemma 21.3** (Gronwall inquality, simple version) *Let $\gamma\colon[0,\infty)\to[0,\infty)$ be Lebesgue integrable on compact intervals and such that, for some $t_0>0$ and $\alpha,\beta\geqslant 0$,*

$$\forall t\leqslant t_0\colon\quad \gamma(t)\leqslant\alpha+\beta\int_0^t\gamma(s)\,\mathrm{d}s \tag{21.15}$$

*Then*

$$\forall t\leqslant t_0\colon\quad \gamma(t)\leqslant\alpha\mathrm{e}^{\beta t} \tag{21.16}$$

*Proof.* Let $\tilde\alpha > \alpha$. Then (21.15) along with continuity of the right-hand side show $\gamma(t) \leqslant \frac{1}{2}(\alpha+\tilde\alpha)$ for $t \geqslant 0$ small. The continuity of $t \mapsto \tilde\alpha e^{\beta t}$ then gives $t_1 := \sup\{t \in [0,t]\colon \gamma(t) \leqslant \tilde\alpha e^{\beta t}\} \in (0,t_0]$. Using this bound in (21.15) shows $\gamma(t) \leqslant \alpha + \tilde\alpha[e^{\beta t}-1]$ for $t \leqslant t_1$ implying, in particular, that $\gamma(t_1) < \tilde\alpha e^{-\beta t_1}$. But that contradicts the definition of $t_1$ unless $t_1 = t_0$, and so $\gamma(t)e^{-\beta t} \leqslant \tilde\alpha$ for all $t \in [0,t_0]$. Taking $\tilde\alpha \downarrow \alpha$ we get (21.16). □

Using this along with $\tau_r \uparrow \tau$ and the Monotone Convergence Theorem we get

$$\forall t \geqslant 0\colon \quad E\Bigl(\sup_{s\leqslant t\wedge\tau}|X_s-\widetilde X_s|^2\Bigr) = \lim_{r\to\infty} g_r(t) \leqslant 3e^{3K^2(t+4)t}E\bigl(|X_0-\widetilde X_0|^2\bigr) \tag{21.17}$$

Setting $C(t) := 3e^{3K^2(t+4)t}$ yields (21.7). For (21.7) we note that $P(X_0 = \widetilde X_0) = 1$ implies $g_r(t) = 0$ for all $r < \infty$. Hence $P(\forall t \leqslant \tau_r\colon X_t = \widetilde X_t) = 1$. Taking $r \to \infty$ then gives $P(\forall t \leqslant \tau\colon X_t = \widetilde X_t) = 1$ as desired. □

Theorem 21.2 expresses the intuitive fact that, as long as a solution is generally unique, it will coincide with the solution for another pair of coefficients in the set where the coefficients coincide provided, of course, that solution is started from the same initial value. This is the statement of *locality* for the solutions of SDEs.

We now claim the following extension of Theorems 20.3 and 21.2:

**Theorem 21.4** *Assume a standard setting for SDE* (20.7) *with initial value $X_0$ and let $D \subseteq \mathbb{R}^d$ be non-empty open and such that*

$$P(X_0 \in D) = 1 \tag{21.18}$$

*Suppose that there are non-empty bounded open sets $\{D_n\}_{n\geqslant1}$ with $\forall n \geqslant 1\colon D_n \subseteq D_{n+1}$ and $D = \bigcup_{n\geqslant1} D_n$ for which the following holds: There is $x_0 \in D_1$ and, for each $n \geqslant 1$, there is $K_n \in (0,\infty)$ such that*

$$\forall t \geqslant 0\,\forall x,y \in D_n\colon \quad \bigl|a(t,x)-a(t,y)\bigr| + \bigl|\sigma(t,x)-\sigma(t,y)\bigr| \leqslant K_n|x-y| \tag{21.19}$$

*and*

$$\forall t \geqslant 0\colon \quad \bigl|a(t,x_0)\bigr| + \bigl|\sigma(t,x_0)\bigr| \leqslant K_n \tag{21.20}$$

*Then there exists an a.s. positive stopping time $T$ and a strong solution $\{X_t\colon t \in [0,T)\}$ to* (20.7) *up to time $T$ with initial value $X_0$ such that, setting*

$$\tau_n := \inf\{t \geqslant 0\colon X_t \notin D_n\} \tag{21.21}$$

*we have*

$$T = \lim_{n\to\infty} \tau_n \tag{21.22}$$

*Moreover, if $\{X_t\colon t \in [0,\widetilde T]\}$ is a strong solution to* (20.7) *up to a finite stopping time $\widetilde T$, then*

$$P(X_0 = \widetilde X_0) = 1 \tag{21.23}$$

*implies*

$$P\Bigl(\forall t \in [0,T\wedge\widetilde T)\colon X_t = \widetilde X_t\Bigr) = 1 \tag{21.24}$$

*In short, $X$ is the unique strong solution until the first "exit" from $D$.*

As we will see in the proof, for the stopping times $\{T_n\}_{n\geqslant 1}$ in Definition 21.1 we can take the sequence $\{\tau_n\}_{n\geqslant 1}$. We write the word "exit" in quotes because $T$ may not really be an exit time from $D$. Indeed, even if $T$ is finite, the solution may simply blow up to infinity at $T$, or oscillate wildly as $t$ increases to $T$ with no limit value at $t = T$. Notwithstanding, $X$ is continuous on $[0, \tau_n]$ whenever $\tau_n < \infty$. It is these considerations that explain why Definition 21.1 needs to be stated as it is. The condition (21.18) is needed to ensure that $T > 0$ a.s.

That Theorem 21.4 extends Theorems 20.3 and 21.2 is seen by setting $D_n := \mathbb{R}^d$ (and thus also $D = \mathbb{R}^d$) for all $n \geqslant 1$ and noting that then $\tau_n = \infty$ for all $n \geqslant 1$, and thus $T = \infty$, by continuity of the solutions. The solution $X$ will not use the values of the coefficients of the equation for the spatial arguments outside $D$ but $\widetilde{X}$ may, and so we keep assuming the standard setting on all of $\mathbb{R}^d$. We bundle the uniqueness clause with the existence of the solution because the uniqueness argument is what actually drives the whole proof.

**21.2 Proof of Theorem 21.4.**

The proof of Theorem 21.4 is based on the argument that is standard in classical ODE theory: Construct local solutions and then, by showing that they must coincide on their common domain, patch these together to get a maximal solution. For the construction of suitable local solutions — which will be those until the first exit from $D_n$ — we will rely on Theorem 20.3 but for this we need to address the problem that a uniform Lipschitz bound on the coefficients $a$ and $\sigma$ applies only on $D_n$. Here we will use:

**Lemma 21.5** (Extension of Lipschitz function) *Let $(\mathscr{X}, \rho)$ be a metric space, $A \subseteq \mathscr{X}$ non-empty and $f\colon A \to \mathbb{R}$ a function such that*

$$\forall x, y \in A\colon \quad \big|f(x) - f(y)\big| \leqslant \rho(x,y) \tag{21.25}$$

*Define*

$$\forall x \in \mathscr{X}\colon \quad h(x) := \sup_{z\in A}\big[f(z) - \rho(x,z)\big] \tag{21.26}$$

*Then $h(x) \in \mathbb{R}$ for all $x \in \mathscr{X}$ and we have*

$$\forall x \in A\colon \quad h(x) = f(x) \tag{21.27}$$

*and*

$$\forall x, y \in \mathscr{X}\colon \quad \big|h(x) - h(y)\big| \leqslant \rho(x,y) \tag{21.28}$$

*Proof.* Since $A \neq \emptyset$, we have $h(x) \in (-\infty, \infty]$. In order to rule out that $h(x)$ is infinite, note that (21.25) along with the triangle inequality imply

$$\forall z, z' \in A\, \forall x \in \mathscr{X}\colon \quad f(z) - \rho(x,z) \leqslant f(z') + \rho(x,z') \tag{21.29}$$

Hence, for each $x \in \mathscr{X}$,

$$h(x) \leqslant \inf_{z\in A}\big[f(z) + \rho(x,z)\big] \tag{21.30}$$

and so $h(x) < \infty$ as well. (The function defined by the supremum could be another candidate for $h$.) Since $h(x) < \infty$, for each $\epsilon > 0$ there is $z_\epsilon \in A$ such that $h(x) \leqslant$

$f(z_\epsilon) - \rho(x, z_\epsilon) + \epsilon$. Then for any $y \in \mathscr{X}$, using $z_\epsilon$ to get an lower bound on $h(y)$ shows

$$\begin{aligned} h(x) - h(y) &\leqslant f(z_\epsilon) - \rho(x, z_\epsilon) + \epsilon - \big[f(z_\epsilon) - \rho(y, z_\epsilon)\big] \\ &= \rho(y, z_\epsilon) - \rho(x, z_\epsilon) + \epsilon \leqslant \rho(x, y) + \epsilon \end{aligned} \tag{21.31}$$

where the triangle inequality was used in the last step. Taking $\epsilon \downarrow 0$ and using symmetry between $x$ and $y$ then shows (21.28).

It remains to prove that $h$ is an extension of $f$. Here the choice $z := x$ for $x \in A$ in (21.26) shows $h(x) \geqslant f(x)$ while taking $z' := x$ in (21.29) and optimizing over $z \in A$ on the left-hand side gives $h(x) \leqslant f(x)$. Hence we get the equality (21.27). $\square$

The key part of the proof of Theorem 21.4 is the content of:

**Lemma 21.6** *Assuming the setting of Theorem 21.4, for each $n \geqslant 1$ there exists a process $\{X_t^{(n)} : t \geqslant 0\}$ such that (1-3) in Definition 21.1 hold for $(X, T)$ replaced by $(X^{(n)}, T_n)$ where*

$$T_n := n \wedge \inf\{t \geqslant 0 \colon X_t^{(n)} \notin D_n\} \tag{21.32}$$

*Moreover, we have*

$$\forall n \geqslant 1\colon \quad P\Big(T_n \leqslant T_{n+1} \,\wedge\, \forall t \in [0, T_n]\colon X_t^{(n+1)} = X_t^{(n)}\Big) = 1 \tag{21.33}$$

*Proof.* For each $t \geqslant 0$ and $x \in \mathbb{R}^d$ define $a^{(n)}(t, x)$ by setting, for each $i = 1, \dots, d$,

$$a_i^{(n)}(t, x) := \sup_{z \in D_n} \big[a_i(t, z) - K_n |x - z|\big] \tag{21.34}$$

Similarly, define $\sigma^{(n)}(t, x)$ by setting, for each $i = 1, \dots, d$ and $j = 1, \dots, m$,

$$\sigma_{ij}^{(n)}(t, x) := \sup_{z \in D_n} \big[\sigma_{ij}(t, z) - K_n |x - z|\big] \tag{21.35}$$

Thanks to Lemma 21.5 and the assumptions (21.19–21.20), $a^{(n)}$ and $\sigma^{(n)}$ now satisfy similar bounds on all of $\mathbb{R}^d$, albeit perhaps with worse constants. Moreover, the continuity of $a(t, \cdot)$ and $\sigma(t, \cdot)$ on $D_n$ permits us to restrict the suprema above to just $z \in D_n \cap \mathbb{Q}^d$ without changing the result. This shows that both $a^{(n)}$ and $\sigma^{(n)}$ are Borel measurable.

Since the boundedness of $D$ ensures that $X_0 1_{\{X_0 \in D\}} \in L^2$, Theorem 20.3 can be applied to construct a strong solution $\{X_t^{(n)} : t \geqslant 0\}$ to SDE (20.7) with coefficients $a^{(n)}$ and $\sigma^{(n)}$ and initial value $X_0 1_{\{X_0 \in D\}}$. Explicitly, $X^{(n)}$ is a continuous adapted process such that

$$\forall t \geqslant 0\colon \quad \int_0^t \big| a^{(n)}(s, X_s^{(n)})\big| \mathrm{d}s < \infty \;\wedge\; \int_0^t \big|\sigma^{(n)}(s, X_s^{(n)})\big|^2 \mathrm{d}s < \infty \quad \text{a.s.} \tag{21.36}$$

and

$$\forall t \geqslant 0 \quad X_t^{(n)} = X_0 1_{\{X_0 \in D\}} + \int_0^t a^{(n)}(s, X_s^{(n)})\, \mathrm{d}s + \int_0^t \sigma^{(n)}(s, X_s^{(n)}) \mathrm{d}B_s \quad \text{a.s.} \tag{21.37}$$

Redefine $X^{(n)}$ on $\{X_0 \notin D_n\}$ by setting

$$X_t^{(n)} := X_0 \quad \text{on } \{X_0 \notin D_n\} \tag{21.38}$$

The known properties of the integral then show that $X^{(n)}$ obeys (21.1–21.2).

The processes $X^{(n)}$ and $X^{(n+1)}$ solve the SDE (20.7) with coefficients that coincide in $D_n$. By Theorem 21.2, these solutions coincide up to the smaller of their first exit times from $D_n$. This gives (21.33) as desired. □

We are now ready to give:

*Proof of Theorem 21.4.* Define $T_n$ by (21.32). On the complement of the union of the implicit null sets in (21.33), set

$$\forall t \in [T_{n-1}, T_n)\colon \quad X_t := X_t^{(n)} \tag{21.39}$$

where $T_0 := 0$ for the sake of this definition, and let $X_t := X_0$ otherwise. Thanks to Lemma 21.6, $X$ then obeys (1-3) in Definition 21.1 for each $n \geqslant 1$. In light of the second part of Definition 21.1 and the fact that $T_n$ coincides with $\tau_n$ from (21.21), this proves the existence part of the claim.

Concerning uniqueness, let $\{\widetilde{X}_t\colon t \in [0, \widetilde{T}]\}$ be another strong solution up to a finite stopping time $\widetilde{T}$ with $\widetilde{X}_0 = X_0$ a.s. Since $a^{(n)} = a$ and $\sigma^{(n)} = \sigma$ on $D_n$,Theorem 21.2 gives

$$P\Big(\forall t \in [0, T_n \wedge \widetilde{T})\colon\ \widetilde{X}_t = X_t^{(n)}\Big) = 1 \tag{21.40}$$

In light of (21.39) and $T_n \uparrow T$, this proves (21.24). □

**21.3 Existence of Bessel processes.**

As a consequence of Theorem 21.2, we conclude:

**Corollary 21.7** *Let $d \in \mathbb{R}$. Then each $X_0 > 0$, the Bessel SDE*

$$\mathrm{d}X_t = \frac{d-1}{2X_t}\mathrm{d}t + \mathrm{d}B_t \tag{21.41}$$

*admits a unique strong solution $\{X_t\colon t \in [0, \tau_0)\}$ where*

$$\tau_0 := \lim_{\epsilon \downarrow 0} \tau_\epsilon \quad \textit{where} \quad \tau_\epsilon := \inf\{t \geqslant 0\colon |X_t| \leqslant \epsilon\} \tag{21.42}$$

*Moreover, if $\tau_0 < \infty$, then this solution extends continuously to $t = \tau_0$ by setting $X_{\tau_0} := 0$.*

*Proof.* The function $x \mapsto \frac{d-1}{2x}$ is uniformly Lipschitz on $D_n := \{x \in \mathbb{R}\colon |x| > 1/n\}$ for each $n \geqslant 1$ and so the solution exists on $D := \bigcup_{n\geqslant 1} D_n = \mathbb{R} \smallsetminus \{0\}$. If $\tau_0 < \infty$ then $X_{\tau_\epsilon} = \epsilon \to 0$ which vanishes in the limit as $\epsilon \downarrow 0$. □

We conclude that *the* $d$-dimensional Bessel process exists uniquely for all $d \in \mathbb{R}$ up to the first hitting time of zero where, of course, uniqueness can be violated. We will see later that SDE techniques are actually not needed for this conclusion, but this is a different part of the story.

Further reading: Karatzas-Shreve, Section 5.2