19. Other PDEs: Poisson, Helmhotz, heat equation

While we focused on the Laplace equation above, similar conclusions can be drawn for a large class of elliptic and parabolic PDEs. In this supplement to the previous lecture, we give an overview of some of the important cases. We will not generally try to *construct* solutions probabilistically; instead, we will mainly contend ourselves with results that *represent* solutions as expectations of Brownian functionals. We only present main results leaving the proofs to homework.

19.1 Elliptic problems.

A natural generalization of the Laplace equation $\Delta u = 0$ — which itself arises in physics as the equation for the electrostatic potential in D with prescribed values on ∂D — is the *Poisson equation*

$$\Delta u = \rho \tag{19.1}$$

This equation governs the electrostatic potential in the situations when electric charge of (signed) density ρ is present inside D. Here we get:

Theorem 19.1 (Poisson equation) *Let $D \subseteq \mathbb{R}^d$ be non-empty and open and let $\rho\colon D \to \mathbb{R}$ be bounded and measurable. Assume that*

$$\forall x \in D\colon \quad P^x(\tau_D < \infty) = 1 \ \wedge \ E^x \int_0^{\tau_D} \big|\rho(B_s)\big| \mathrm{d}s < \infty \tag{19.2}$$

Then any $u \in C^2(D) \cap C(\overline{D})$ such that

$$\forall x \in D\colon \quad \Delta u(x) = \rho(x) \tag{19.3}$$

obeys

$$\forall x \in D\colon \quad u(x) = E^x\big(u(B_{\tau_D})\big) - \frac{1}{2} E^x \int_0^{\tau_D} \rho(B_s) \mathrm{d}s \tag{19.4}$$

The proof is a fairly straightforward exercise in martingale theory. The factor "1/2" in (19.4) arises from the similar factor in Itô formula. Note that condition (19.2) holds when

$$\rho \in L^\infty(D) \ \wedge \ \forall x \in D\colon \quad E^x \tau_D < \infty \tag{19.5}$$

Under this condition one can check that the function defined by the right-hand side of (19.4) for any prescribed (bounded measurable) boundary values of u solves (19.3) at all Lebesgue-differentiability points of ρ (which is a set of full Lebesgue measure). Since ϱ is assumed bounded, once we know that

$$\forall x \in \partial D\colon \quad \lim_{\substack{z \to x \\ z \in D}} E^z \tau_D = 0, \tag{19.6}$$

the continuity of u at the boundary is handled just as for the Laplace equation. For D bounded with all boundary points regular, the condition (19.6) is checked from the fact that $\{\tau_{-z+D}\colon z \in D\}$ is uniformly integrable under P^0.

Another way to generalize the Laplace equation is by adding a linear term in u. We treat the inhomogeneous case which subsumes the Poisson equation as well:

Theorem 19.2 (Helmholtz equation) *Let $D \subseteq \mathbb{R}^d$ be non-empty and open and, given $\kappa \in \mathbb{R}$ and a bounded measurable $\rho\colon D \to \mathbb{R}$, assume $u \in C^2(D) \cap C(\overline{D})$ obeys*

$$\forall x \in D\colon \quad \tfrac{1}{2}\Delta u(x) + \kappa u(x) = \rho(x) \tag{19.7}$$

Assuming $\forall x \in D\colon\ E^x \tau_D < \infty$ and, when $\kappa > 0$, also

$$\forall x \in D\colon \quad E^x \mathrm{e}^{\kappa \tau_D} < \infty \tag{19.8}$$

we then have

$$\forall x \in D\colon \quad u(x) = E^x\big(u(B_{\tau_D})\mathrm{e}^{\kappa\tau_D}\big) - E^x \int_0^{\tau_D} \rho(B_s)\mathrm{d}s \tag{19.9}$$

The Helmholtz equation appears in PDE literature when trying to solve the *wave equation* by separation of variables, i.e., using functions that factor into a product of a time-dependent term and a space-dependent term. For $\kappa \leqslant 0$, the above may be interpreted using Brownian motion that is *killed* or *annihilated* — which means "moved to a special 'cemetery' state" — at an independent exponential time with parameter $-\kappa$. Note that this is different from the Brownian motion $\{B_{t\wedge\tau_D}\colon t \geqslant 0\}$ which is stopped, or *absorbed*, at the boundary.

The representation of the kind (19.9) applies even to the situation when κ is allowed to depend on the spatial variable. This includes the following problem.

Lemma 19.3 (Eigenfunction representation) *Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be measurable, bounded from below and let $u \in C^2(\mathbb{R}^d)$ be a function such that, for some $\lambda \in \mathbb{R}$,*

$$\forall x \in \mathbb{R}^d\colon \quad -\tfrac{1}{2}\Delta u(x) + V(x)u(x) = \lambda u(x) \tag{19.10}$$

Then for any $t \geqslant 0$,

$$\forall x \in D\colon \quad u(x) = E^x\left(u(B_t)\exp\left\{\int_0^t [\lambda - V(B_s)]\mathrm{d}s\right\}\right) \tag{19.11}$$

In order to explain the title of the lemma, a function satisfying (19.10) is called an *eigenfunction* of the operator $-\frac{1}{2}\Delta + V(\cdot)$ with *eigenvalue* λ. (The formal concept requires also that u be an element of a suitable Hilbert space such as $L^2(\mathbb{R}^d)$.) Operators of this form play a very important role in quantum mechanics and the representation (19.11) can sometimes be useful for the analysis of their eigenfunctions and eigenvalues.

19.2 Parabolic problems.

Another equation that can be represented probabilistically is the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u \tag{19.12}$$

We will treat the inhomogeneous case as that poses no extra difficulty:

Theorem 19.4 (heat equation, Cauchy problem) *Let $D \subseteq \mathbb{R}^d$ be non-empty bounded and open. Let $\varrho\colon D \to \mathbb{R}$ be bounded measurable with*

$$\forall x \in D\, \forall t \geqslant 0\colon \quad E^x \int_0^{t\wedge\tau_D} \big|\rho(B_s)\big|\mathrm{d}s < \infty \tag{19.13}$$

Assume $u\colon [0,\infty)\times\overline{D}\to\mathbb{R}$ *is bounded continuous, of type* $C^{1,2}((0,\infty)\times D)$*, and such that*

$$\forall t>0\,\forall x\in D\colon\quad \frac{\partial u}{\partial t}(t,x)=\tfrac12\Delta u(t,x)+\rho(x) \tag{19.14}$$

and

$$\forall x\in\partial D\,\forall t\geqslant 0\colon\quad u(t,x)=0 \tag{19.15}$$

Then

$$u(t,x)=E^x\Bigl(u(0,B_t)1_{\{\tau_D>t\}}\Bigr)+E^x\Bigl(\int_0^{t\wedge\tau_D}\rho(B_s)\mathrm{d}s\Bigr) \tag{19.16}$$

The result becomes particularly clean when $D=\mathbb{R}^d$, $\rho=0$ and u is thus a bounded solution to the Cauchy problem for the full-space heat equation (19.12). In this case $\tau_D=\infty$ and no assumption of the kind (19.15) is necessary and so

$$u(t,x)=E^x\bigl(u(0,B_t)\bigr) \tag{19.17}$$

Inspired by this, we note:

Lemma 19.5 (Full-space heat equation) *For any bounded measurable* $f\colon\mathbb{R}^d\to\mathbb{R}$*, set*

$$\forall t\geqslant 0\,\forall x\in\mathbb{R}^d\colon\quad u_f(t,x):=E^x\bigl(f(B_t)\bigr) \tag{19.18}$$

Then u_f *is of type* $C^{1,2}((0,\infty)\times D)$*, solves* (19.12) *and obeys*

$$\lim_{t\downarrow 0}u(t,x)=f(x) \tag{19.19}$$

at all Lebesgue-differentiability points of f*. If also* $f\in C(\mathbb{R})$ *then* $u\in C([0,\infty)\times\mathbb{R})$*.*

Note that (19.19) only "probes" the so called radial limit, which in our case means that the spatial coordinate remains constant. This is what needs to be varied as well to get the continuity of u_f when f is continuous.

The representation (19.18) is classical in analysis and it requires no probability because it amounts to a convolution with the probability density of $\mathcal{N}(x,t)$. Matters get more interesting in non-trivial domains D. Here if (19.15) is not assumed, then (19.16) is replaced by

$$u(t,x)=E^x\Bigl(u\bigl((t-\tau_D)\vee 0,B_{t\wedge\tau_D}\bigr)\Bigr)+E^x\int_0^{t\wedge\tau_D}\rho(B_s)\mathrm{d}s \tag{19.20}$$

For the homogeneous problem this readily yields:

Corollary 19.6 (Maximum principle for heat equation) *Let* $T>0$ *and* $D\subseteq\mathbb{R}^d$ *be non-empty, bounded and open. If* $u\in C([0,T]\times\overline{D})$ *is of type* $C^1((0,T))\times C^2(D)$ *and solves* (19.12) *for all* $(t,x)\in(0,T)\times D$*, then*

$$\sup_{x\in(0,T)\times D}\bigl|u(t,x)\bigr|\leqslant\sup_{x\in\partial([0,T]\times\overline{D})}\bigl|u(t,x)\bigr| \tag{19.21}$$

In particular, if $u(0,\cdot)=0$ *on* D *and* $u(\cdot,x)=0$ *on* $(0,T)$ *for all* $x\in\partial D$*, then* $u=0$*. The mixed Cauchy/boundary-value problem for the heat equation in* $[0,T]\times D$ *has a unique solution.*

The Maximum Principle resolves uniqueness for the heat equation in bounded domains. A growth condition of sorts needs to be imposed in unbounded domains. Here is a theorem whose proof, which is somewhat delicate, can be found in Karatzas-Shreve:

Theorem 19.7 (Tychonoff's Uniqueness theorem) *For any $T > 0$ and any function $u \in C^{1,2}((0,T) \times \mathbb{R})$ that solves* (19.12) *in* $(0,\infty) \times \mathbb{R}$,

$$\forall x \in \mathbb{R}\colon \quad \lim_{\substack{(t,y)\to(0,x)\\ (t,y)\in(0,\infty)\times\mathbb{R}}} u(t,y) = 0 \tag{19.22}$$

and

$$\exists C, c > 0\, \forall x \in \mathbb{R}\colon \sup_{0<t<T} \bigl|u(t,x)\bigr| \leqslant C e^{cx^2} \tag{19.23}$$

imply $u = 0$.

Our final result concerns the same generalization that the Helmhotz equation and the eigenfunction representation were relative to the Laplace equation. For simplicity, we treat just the full-space homogeneous case this time.

Theorem 19.8 (Feynman-Kac formula) *Let $u \in C^{1,2}((0,\infty) \times \mathbb{R}^d) \cap C([0,\infty) \times \mathbb{R}^d)$ be bounded and such that, for some bounded measurable $V\colon \mathbb{R}^d \to \mathbb{R}$,*

$$\forall t > 0\, \forall x \in \mathbb{R}^d\colon \quad \frac{\partial u}{\partial t}(t,x) = \tfrac{1}{2}\Delta u(t,x) + V(x)u(t,x) \tag{19.24}$$

Then

$$u(t,x) = E^x\biggl(u(0,B_t)\exp\Bigl\{\int_0^t V(B_s)\mathrm{d}s\Bigr\}\biggr) \tag{19.25}$$

The reason for R. Feynman's name appearing on this is that formulas of this kind are the basis of his "path integral" representation of solutions to the Schrödinger equation and other problems in quantum mechanics. It was really M. Kac who derived the above formula in the context of the heat-equation where the "path integral" acquires the rigorous meaning of integral with respect to the Wiener measure.

The original derivation of the above formula was done in the context of *backward* heat equation, i.e.,

$$-\frac{\partial u}{\partial t} = \tfrac{1}{2}\Delta u + V(\cdot)u \tag{19.26}$$

Here one gets, for $t \in [0,T]$ and $x \in \mathbb{R}^d$,

$$u(t,x) = E^x\biggl(u(T,B_{T-t})\exp\Bigl\{\int_0^{T-t} V(B_s)\mathrm{d}s\Bigr\}\biggr) \tag{19.27}$$

i.e., the solution is represented by its *terminal* condition at time $t = T$.

Further reading: Karatzas-Shreve, Sections 4.3-4.3