18. Boundary regularity

In this lecture we wrap up the discussion of the Dirichlet problem for the Laplace equation started in the previous lecture.

18.1 Boundary points of continuity.

Recall that in Proposition 17.4 we showed that, for any $g\colon \partial D \to \mathbb{R}$ bounded and measurable, the function $u_g\colon \overline{D} \to \mathbb{R}$ given by

$$u_g(x) := E^x\bigl(g(B_{\tau_D})1_{\{\tau_D<\infty\}}\bigr) \tag{18.1}$$

is $C^2(D)$ and solves the Laplace equation (17.2). Note that while $u_g = g$ on ∂D, this does not say anything about the limit behavior of $u_g(z)$ as $z \to x \in \partial D$ nor about the relation of the potential limit to $g(x)$. As

$$P^x(\tau_D = \infty) = 1 - u_1(x) \tag{18.2}$$

the same applies to the second term in (17.15).

Building on the observations in item (2) of the list at the beginning of Section 17.2, we now readily check that continuity at boundary points is not generally guaranteed. The advantage of the probabilistic representation (18.1) over potential-theoretic tools is that it makes this very easy to demonstrate.

Starting with two-dimensional situations, let $D := \{x \in \mathbb{R}^d\colon 0 < |x| < 1\}$ be the punctured unit disc. Then $\partial D = \{0\} \cup \{x \in \mathbb{R}^d\colon |x| = 1\}$ and Corollary 16.6 tells us that $B_{\tau_D} \notin 0$ a.s. It follows that the value $g(0)$ is immaterial for (18.1). Two alternatives are possible — either the limit of $u_g(z)$ as $z \to 0$ in D does not exist, or it does but it is unrelated to what we set for $g(0)$ — each of which imply that the Dirichlet problem is not solvable for generic (bounded measurable) g.

A similar reasoning applies in all $d \geqslant 3$ where removing a one-dimensional line segment from a unit ball creates part of the boundary that will not be found by Brownian motion (as its projection on orthogonal subspace of the segment is a $d-1 \geqslant 2$-dimensional Brownian motion) and so we are not free to prescribe boundary values on it. In this case D is even connected. These examples are readily generalized as follows: Recall that the α-dimensional Hausdorff outer measure $\mathcal{H}^\alpha(A)$ of $A \subseteq \mathbb{R}^d$ is defined as follows:

$$\mathcal{H}^\alpha(A) := \inf_{\epsilon>0} \inf\Bigl\{\sum_{i\geqslant1} \operatorname{diam}(A_i)^\alpha\colon A \subseteq \bigcup_{i\geqslant1} A_i \,\wedge\, \forall i \geqslant 1\colon \operatorname{diam}(A_i) < \epsilon\Bigr\} \tag{18.3}$$

where diam is the Euclidean diameter. The following lemma is left as an exercise:

Lemma 18.1 *Let $d \geqslant 3$ and let $D \subseteq \mathbb{R}^d$ be non-empty, bounded and open. Then*

$$\forall A \in \mathcal{B}(\partial D)\colon \quad \mathcal{H}^{d-2}(A) = 0 \quad \Rightarrow \quad \forall x \in D\colon\ P^x(B_{\tau_D} \in A) = 0 \tag{18.4}$$

This shows that, in $d \geqslant 3$, any subset of ∂D that has Hausdorff dimension less than $d-2$ will not be hit by Brownian motion and so g can be changed arbitrarily there without affecting (18.1). However, as pointed out already in Section 17.2, a more subtle obstruction to boundary continuity arises from that fact that Brownian motion started near a boundary point $x \in \partial D$ does not hit ∂D near x — and so u_g samples values of g that can be very different from $g(x)$. As it turns out, this can ultimately be phrased as a statement for Brownian motion started directly *from* the boundary point.

Theorem 18.2 *Let $D \subseteq \mathbb{R}^d$ be non-empty open and, writing $\{B_t \colon t \geqslant 0\}$ for a d-dimensional standard Brownian motion, set*

$$\hat\tau_D := \inf\{t > 0 \colon B_t \notin D\}. \tag{18.5}$$

For each $g \colon \partial D \to \mathbb{R}$ bounded measurable, let $u_g \colon \overline{D} \to \mathbb{R}$ be as in (18.1)*. Then for any $x \in \partial D$, the following are equivalent:*

(1) $P^x(\hat\tau_D = 0) = 1$

(2) *for each bounded measurable $g \colon \partial D \to \mathbb{R}$:*

$$g \text{ is continuous at } x \quad \Rightarrow \quad u_g \text{ is continuous at } x \tag{18.6}$$

Here P^x is the law of B such that $P^x(B_0 = x) = 1$.

The reader might wonder whether $\hat\tau_D$ is in fact a well defined random variable. This follows from the fact that

$$\hat\tau_D := \inf_{n\in\mathbb{N}} \sup\bigl\{k2^{-n} \colon k \in \mathbb{N} \smallsetminus \{0\} \wedge (\forall \ell \leqslant k \colon B_{\ell 2^{-n}} \in D)\bigr\} \tag{18.7}$$

where the infimum is actually an $n \to \infty$ limit because the supremum is non-increasing in n. However, $\hat\tau_D$ is not a stopping time unless the underlying filtration is right-continuous because $\{\hat\tau_D \leqslant t\}$ requires an "infinitesimal peek" at the Brownian path past time t.

Notwithstanding, $\hat\tau_D$ is a stopping time for the filtration $\mathcal{F}^B_{t+}\}_{t\geqslant 0}$ and $\{\hat\tau_D = 0\} \in \mathcal{F}^B_{0+}$-measurable. Blumenthal's Zero-One Law therefore gives

$$\forall x \in \partial D \colon \quad P^x(\hat\tau_D = 0) \in \{0, 1\} \tag{18.8}$$

Thus either (almost) all Brownian paths started from infinitesimally close to x exit D instantaneously, or (almost) none does so.

18.2 Proof of Theorem 18.2.

We start with a key technical step:

Proposition 18.3 *Let $D \subseteq \mathbb{R}^d$ be open and non-empty. Then, for each $x \in \partial D$,*

(1) *for all $\{z_n\}_{n\in\mathbb{N}} \in D^{\mathbb{N}}$ satisfying $z_n \to x$ and all $r > 0$,*

$$\liminf_{n\to\infty} P^{z_n}\bigl(\tau_D < \infty \wedge |B_{\tau_D} - x| < r\bigr) \geqslant P^x(\hat\tau_D = 0) \tag{18.9}$$

(2) *there exists $\{z_n\}_{n\in\mathbb{N}} \in D^{\mathbb{N}}$ with $z_n \to x$ such that*

$$\lim_{r\downarrow 0} \limsup_{n\to\infty} P^{z_n}\bigl(\tau_D < \infty \wedge |B_{\tau_D} - x| < r\bigr) = P^x(\hat\tau_D = 0) \tag{18.10}$$

To see why these are relevant, we first give:

Proof of Theorem 18.2 from Proposition 18.3. Let $g \colon \partial D \to \mathbb{R}$ be bounded and measurable and $x \in \partial D$ such that $P^x(\hat\tau_D > 0) = 0$. Write

$$E_r := \{\tau_D = \infty\} \cup \bigl\{\tau_D < \infty \wedge |B_{\tau_D} - x| \geqslant r\bigr\} \tag{18.11}$$

for the event complementary to that in (18.9) and (18.10). For $z \in D$ we then have

$$\begin{aligned} \bigl|u_g(z)-g(x)\bigr| &\leqslant \Bigl|E^z\bigl(g(B_{\tau_D})1_{\{\tau_D<\infty\}}-g(x)\bigr)\Bigr| \\ &\leqslant \sup_{y\in B(x,r)\cap\partial D}\bigl|g(y)-g(x)\bigr|+2\|g\|_\infty P^z(E_r) \end{aligned} \tag{18.12}$$

Proposition 18.3(1) along with the continuity of g at x show that this tends to zero in the limit as $z \to x$ in D followed by $r \downarrow 0$. Since $u_g(x) = g(x)$, this implies continuity of u_g at x for all $a \in \mathbb{R}$ and all g as above.

For the opposite direction assume that $x \in \partial D$ obeys $P^x(\hat\tau_D > 0) > 0$. Proposition 18.3(2) gives $\delta > 0$, $r > 0$ and a sequence $\{z_n\}_{n\geqslant 1} \subseteq D$ with $z_n \to x$ such that

$$\forall n \geqslant 1\colon \quad P^{z_n}(E_r) \geqslant \delta \tag{18.13}$$

Setting $h(x) := u_g(z) + P^z(\tau_D = \infty)$ for

$$g(z) := \max\{1, r^{-1}|z-x|\}, \tag{18.14}$$

use that $g \geqslant 0$ and $g(z) = 1$ when $|z-x| \geqslant r$ we get

$$h(z_n) \geqslant E^{z_n}\bigl(g(B_{\tau_D})1_{\{|B_{\tau_D}-x|\geqslant r\}}\bigr) + P^{z_n}(\tau_D = \infty) \geqslant P^{z_n}(E_r) \geqslant \delta > 0 \tag{18.15}$$

As $h_(x) = g(x) = 0$, the function h fails to be continuous at x. But this means that either u_g fails to be continuous or $u_1(z) := E^z(1 \cdot 1_{\{\tau_D<\infty\}}) = 1 - P^z(\tau_D = \infty)$ fails to be continuous. $\square$

It remains to prove Proposition 18.3. The proof of (1) needs:

Lemma 18.4 *Let $t > 0$ and $A \in \sigma(B_s\colon s \geqslant t)$. Then $x \mapsto P^x(A)$ is continuous.*

Proof. Realizing the problem on the Wiener space, let $\theta_t(A) := \{\omega(\cdot - t)\colon \omega \in A\}$. Since $B_t = \mathcal{N}(x, t1)$ under P^x, where 1 is the d-dimensional identity matrix, writing g_t for the density of $\mathcal{N}(x, t1)$ and conditioning on B_t yields

$$P^x(A) = \int_{\mathbb{R}^d} g_t(z-x) P^z\bigl(\theta_t(A)\bigr)\mathrm{d}z \tag{18.16}$$

The claim then follows from the fact that $z \mapsto g_t(z)$ is continuous in $L^1(\mathbb{R}^d)$. $\square$

With this in hand, we then give:

Proof of Proposition 18.3(1). Fix $r > 0$ and recall the event E_r from (18.11). Then write, for any $\epsilon > 0$,

$$P^z(E_r) \leqslant P^z(\tau_D > \epsilon) + P^z\bigl(\tau_D \leqslant \epsilon \,\wedge\, |B_{\tau_D} - x| \geqslant r\bigr) \tag{18.17}$$

Assuming $|z-x| < r/2$, the second probability is bounded by

$$P^z\Bigl(\sup_{s\leqslant\epsilon}|B_s - z| > r/2\Bigr) = P^0\Bigl(\sup_{s\leqslant 1}|B_s| > r\epsilon^{-1/2}/2\Bigr) \tag{18.18}$$

where we also invoked shift invariance and diffusive scaling of the Brownian motion (see Proposition 5.1(1,2)). The probability on the right-hand side vanishes as $\epsilon \downarrow 0$ by continuity of the paths.

Concerning the first term on the right of (18.17), pick $\delta \in (0, \epsilon)$ and let

$$\tau_{D,\delta} := \inf\{t \geqslant \delta \colon B_t \notin D\} \tag{18.19}$$

Then $\tau_D \leqslant \tau_{D,\delta}$ and so

$$P^z(\tau_D > \epsilon) \leqslant P^z(\tau_{D,\delta} > \epsilon) \tag{18.20}$$

The reason for invoking $\tau_{D,\delta}$ is that $\{\tau_{D,\delta} > \epsilon\} \in \sigma(B_s \colon s \geqslant \delta)$. Lemma 18.4 thus shows

$$\limsup_{n\to\infty} P^{z_n}(\tau_D > \epsilon) \leqslant P^x(\tau_{D,\delta} > \epsilon) \tag{18.21}$$

whenever $z_n \to x$ inside D. We now observe that $\tau_{D,\delta} \downarrow \hat\tau_D$ as $\delta \downarrow 0$. This shows

$$\lim_{\epsilon\downarrow 0} \limsup_{\delta\downarrow 0} P^x(\tau_{D,\delta} > \epsilon) \leqslant \lim_{\epsilon\downarrow 0} P^x(\hat\tau_D \geqslant \epsilon) = P^x(\hat\tau_D > 0) \tag{18.22}$$

thus proving (18.9). □

The proof of the second part turns out to be more involved:

Proof of Proposition 18.3(2). Note that we may assume that $P^x(\hat\tau_D > 0) > 0$ and, by Blumenthal's zero-one law, thus $P^x(\hat\tau_D > 0) = 1$ for otherwise the claim follows from (1) and the fact that the left-hand side of (18.10) is at most one. For the same reason we may assume that $d \geqslant 2$ because, in $d = 1$, we have $P^x(\hat\tau_D = 0) = 1$ by the fact that the set of times when the Brownian motion returns its starting point have no points isolated from the right. We will nonetheless continue writing $P^x(\hat\tau_D = 0)$ until the fact that this is zero is actually needed.

We start with some observations. Abbreviate

$$F_r := \bigl\{\hat\tau_D < \infty \ \wedge \ |B_{\hat\tau_D} - x| < r\bigr\} \tag{18.23}$$

and note that $P^{z_n}(F_r)$ coincides with probability in (18.10) because, by continuity of paths, $P^z(\hat\tau_D = \tau_D) = 1$ for all $z \in D$. This event is relevant because

$$\bigcap_{r>0} F_r = \bigl\{\hat\tau_D < \infty \ \wedge \ B_{\hat\tau_D} = x\bigr\} \tag{18.24}$$

implies

$$\bigcap_{r>0} F_r \supseteq \{\hat\tau_D = 0\} \cap \{B_0 = x\} \tag{18.25}$$

and

$$\begin{aligned} \{B_0 = x\} \cap \bigcap_{r>0} F_r \ &\subseteq \{\hat\tau_D = 0\} \cap \{B_0 = x\} \\ &\cup \Bigl(\{B_0 = x\} \cap \bigl\{\inf\{t > 0 \colon B_t = x\} < \infty\bigr\}\Bigr). \end{aligned} \tag{18.26}$$

By lack of recurrence of Brownian motion to points in all $d \geqslant 2$ (see Corollary 16.6), the event on the right has P^x-probability zero and so we get

$$\lim_{r\downarrow 0} P^x(F_r) = P^x\Bigl(\bigcap_{r>0} F_r\Bigr) = P^x(\hat\tau_D = 0) \tag{18.27}$$

by continuity of probability and the fact that $r \mapsto F_r$ is non-decreasing.

Since $\{\hat\tau_D \leqslant t\} \in \mathcal{F}_s$ for each $s > t$ by, e.g., (18.7), $\hat\tau_D$ is a stopping time for the filtration $\{\mathcal{F}_{t+}\}_{t\geqslant 0}$. Relying on the same argument as (17.27), the Strong Markov Property gives

$$E^x(1_{F_r}|\mathcal{F}_{\hat\tau_D\wedge t+}) = P^{B_{\hat\tau_D\wedge t}}(F_r) \quad P^x\text{-a.s.} \tag{18.28}$$

The Backward Lévy Theorem then shows

$$E^x(1_{F_r}|\mathcal{F}_{\hat\tau_D\wedge t+}) \underset{t\downarrow 0}{\longrightarrow} E^x\Bigl(1_{F_r}\Bigm|\bigcap_{t>0}\mathcal{F}_{\hat\tau_D\wedge t+}\Bigr) = P^x(F_r) \quad P^x\text{-a.s.} \tag{18.29}$$

where we used that $\bigcap_{t>0}\mathcal{F}_{\hat\tau_D\wedge t+} \subseteq \mathcal{F}_{0+}$ and that, by Blumenthal's Zero-One Law, P^x is trivial on $\mathcal{F}_{0+}$.

Combining the above observations along with the assumption that $\hat\tau_D > 0$ P^x-a.s., we have shown that

$$\forall r > 0\colon\ P^{B_t}(F_r) \underset{t\downarrow 0}{\longrightarrow} P^x(F_r) \quad P^x\text{-a.s.} \tag{18.30}$$

where the null set may depend on r. Using that a countable interesection of full measure events is a full measure event, it follows that there exists a Brownian path B such that $\hat\tau_D > 0$ and the limit (18.30) takes place for all $r \in \{2^{-k}\colon k \geqslant 0\}$ simultaneously. For the choice $z_n := B_{1/n}$ when $n \geqslant n_0 := \lceil 1/\hat\tau_D\rceil$ and $z_n := z_{n_0}$ otherwise we then get that, thanks to the monotonicity of $r \mapsto F_r$,

$$\begin{aligned}\lim_{r\downarrow 0}\,\limsup_{n\to\infty} P^{z_n}\bigl(\tau_D < \infty\ \wedge\ |B_{\tau_D} - x| < r\bigr)\\ = \lim_{k\to\infty}\,\limsup_{n\to\infty} P^{B_{1/n}}(F_{2^{-k}}) = \lim_{k\to\infty} P^x(F_{2^{-k}}) = P^x(\hat\tau_0 = 0)\end{aligned} \tag{18.31}$$

thus proving the second part of the claim as well. □

Note that equality may *not* hold in (18.9) for all sequences when $P^x(\hat\tau_D = 0)$ is less than one, and thus vanishes. Indeed, taking $\{z_n\}_{n\geqslant 1} \subseteq D$ for which $z_n \to x$ but that so while nearly "grazing" ∂D — which means $\mathrm{dist}(z_n, \partial D) \ll \mathrm{dist}(z_n, x)$ — effectively probes other parts of the boundary rather than a neighborhood of x. The argument after (18.30) shows that this is not the case for sequences derived from Brownian paths started from x and contained in $\{\hat\tau_D > 0\}$ and, for these, (18.10) then does hold.

18.3 Regular points.

In light of Theorem 18.2, it is natural to put forward:

Definition 18.5 *We say that $x \in \partial D$ is regular if*

$$P^x(\hat\tau_D = 0) = 1 \tag{18.32}$$

The salient conclusion for solving the Laplace equation with Dirichlet boundary condition is then stated as:

Corollary 18.6 *For all $D \subseteq \mathbb{R}^d$ non-empty open, the Dirichlet problem (17.2–17.3) is solvable in the class of bounded functions for all bounded boundary conditions $g \in C(\partial D)$ if and only if every $x \in \partial D$ is regular.*

Proof. That u_g is harmonic on D for each bounded $g \in C(D)$ has been checked in the proof of Proposition 17.4. Equivalence of the continuity at the boundary with regularity thereof is then verified using Theorem 18.2. □

We note that the concept of a regular point originates in potential theory where it is typically *defined* by alternative (1) in Theorem 18.2. A necessary and sufficient condition for a point $x \in \partial D$ to be regular is the *barrier condition*. This refers to the existence of a function $f \in C(\overline{D})$ that is superharmonic on D (i.e., with mean value on each $B(z,r) \subseteq D$ no larger than $f(z)$) such that $f > 0$ on $\overline{D} \smallsetminus \{x\}$ and $f(x) = 0$.

In $D \subseteq \mathbb{R}^2$ with ∂D connected and containing at least two points, all boundary points are regular while in $d \geqslant 3$, the tip of a sufficiently thin inward "spike" — known as the *Lebesgue thorn*, as such examples were first noted by Lebesgue in 1912 — is not regular.

To prevent such situations from happening, one may impose a *cone condition* on ∂D which states that if $B(x,r) \cap D^{\mathrm{c}}$ contains (for some small $r > 0$) a cone of positive aperture with apex at $x \in \partial D$, then x is regular. The proof of this relies on two convenient features of this concept:

(1) *locality* of the event $\{\hat{\tau}_D = 0\}$, which means irrelevance of changes to D outside a ball of any positive radius centered at the boundary point, and
(2) *monotonicity* of $D \mapsto \hat{\tau}_D$ under the set inclusion.

These show that, if a Brownian motion started from the apex of any cone of positive aperture enters the cone instantaneously, then it exits D instantaneously at any $x \in \partial D$ to which such a cone can be attached inside D^{c}.

Thanks to Theorem 18.2 we know that, assuming all points in ∂D to be regular, the one-parameter family of functions in Proposition 17.4 solve the Dirichlet problem (17.2–17.3). An intriguing question is whether other (linearly independent) bounded harmonic functions can be added to this that vanish on ∂D. This is resolved by way of compactification of D by adding to D a so-called *Poisson boundary* at infinity which then allows us to treat this again as a Dirichlet problem in a compact domain.

Further reading: Karatzas-Shreve, Sections 4.2C