17. Solving Dirichlet-Laplace problem

We proceed to discuss some specific applications of the Itô formula to solutions of elliptic and parabolic differential equations. Continuing on with on our recent practice, throughout we write P^x for the law of an underlying process X — typically, the d-dimensional standard Brownian motion — started at x in the sense

$$P^x(X_0 = x) = 1 \tag{17.1}$$

and denote by E^x the expectation with respect to P^x.

17.1 Dirichlet problem.

Recall the notation Δu for the Laplacian of a (twice-differentiable) function u. We start with a classical problem from PDE theory.

Definition 17.1 (Dirichlet problem for Laplacian equation) *Given $D \subseteq \mathbb{R}^d$ non-empty open and $g\colon \partial D \to \mathbb{R}$, we say that $u\colon D \to \mathbb{R}$ solves the* Dirichlet problem *for Laplacian equation with boundary condition g if $u \in C^2(D)$,*

$$\forall x \in D\colon \quad \Delta u(x) = 0 \tag{17.2}$$

and

$$\forall x \in \partial D\colon \quad \lim_{\substack{y\to x\\ y\in D}} u(y) = g(x) \tag{17.3}$$

A function satisfying (17.2) called *harmonic*, although the definition of that concept is usually done via the *Mean-Value Property* that reads

$$u(x) = \frac{1}{\lambda(B(0,r))}\int_{B(0,r)} u(x+z)\lambda(\mathrm{d}z) \tag{17.4}$$

whenever $x \in D$ and $B(x,r) := \{y \in \mathbb{R}^d\colon |x-y| < r\}$ obeys $\overline{B(x,r)} \subseteq D$. The advantage of this definition is that it only requires u to be locally Lebesgue integrable. That this implies $u \in C^2(D)$ and (17.5) needs an argument for which we refer to PDE literature.

As is readily checked, (17.3) forces g to be continuous on ∂D (under the Euclidean metric) and so the above in fact reduces to finding $u \in C^2(D) \cap C(\overline{D})$ such that $\Delta u = 0$ on D and $u = g$ on ∂D, for a given $g \in C(\partial D)$. Our starting observation is that any *bounded* solution to this problem can be represented probabilistically in a fairly large class of underlying domains:

Theorem 17.2 (Kakutani 1944) *Let $D \subseteq \mathbb{R}^d$ be a non-empty open set and let*

$$\tau_D := \inf\{t \geqslant 0\colon B_t \notin D\} \tag{17.5}$$

the first exit time from D of standard d-dimensional Brownian motion B. Then τ_D is a stopping time with $B_{\tau_D} \in \partial D$ on $\{\tau_D < \infty\}$. Moreover, if $u \in C^2(D) \cap C(\overline{D})$ is bounded and such that $\Delta u = 0$ on D, then

$$\forall x \in D\colon \quad P^x(\tau_D < \infty) = 1 \tag{17.6}$$

implies

$$\forall x \in D\colon \quad u(x) = E^x\big(u(B_{\tau_D})\big). \tag{17.7}$$

Proof. Let $D_n := \{x \in D\colon \operatorname{dist}(x, D^c) > 1/n \ \wedge\ \operatorname{dist}(0,x) < n\}$ where "dist" refers to the Euclidean distance. Assume that n is so large that $D_n \neq \emptyset$. Then

$$\sup_{x \in D_n} \bigl|\nabla u(x)\bigr| < \infty \tag{17.8}$$

and the Itô formula cast in vector notation reads

$$\mathrm{d}u(B_{t\wedge\tau_{D_n}}) = 1_{\{\tau_{D_n}>t\}} \nabla u(B_{t\wedge\tau_{D_n}}) \cdot \mathrm{d}B_t + 1_{\{\tau_{D_n}>t\}} \Delta u(B_{t\wedge\tau_{D_n}})\mathrm{d}t \tag{17.9}$$

where condition (17.8) is needed to define the stochastic integral. The second term exists trivially since, in light of $\Delta u = 0$ on D, it vanishes. It follows that

$$M_t := u(B_{t\wedge\tau_{D_n}}) \tag{17.10}$$

is a continuous local martingale which, since u is bounded, is a bounded martingale under P^x for any $x \in D$. Hence,

$$\forall t \geqslant 0\, \forall x \in D\colon \quad u(x) = E^x(M_0) = E^x(M_t) = E^x\bigl(u(B_{t\wedge\tau_{D_n}})\bigr) \tag{17.11}$$

Now use that, from (17.6), $t \wedge \tau_{D_n} \to \tau_D < \infty$ as $n \to \infty$ followed by $t \to \infty$ with $B_{t\wedge\tau_{D_n}} \to B_{\tau_D} \in \partial D$ in these limits, P^x-a.s., by continuity of B. As u is bounded and extends continuously to $\overline{D}$, the double limit passes into the expectation using the Bounded Convergence Theorem. □

Note that (17.7) expresses u in D using its boundary values. Another way to write this is via

$$u(x) = \int_{\partial D} u(z) H^D(x, \mathrm{d}z) \tag{17.12}$$

where $H^D\colon D \times \mathcal{B}(\mathbb{R}^d) \to [0,1]$ is given by

$$H^D(x, A) := P^x(B_{\tau_D} \in A). \tag{17.13}$$

In short, $H^D(x,\cdot)$ is the *exit distribution* from D of the Brownian motion started at x. In analysis and PDE theory, H^D is called the *harmonic measure* because, as can be shown, $x \mapsto H^D(x, A)$ is harmonic on D for each $A \in \mathcal{B}(\mathbb{R}^d)$. The representation (17.12) is the basis of various *Poisson integral formulas*.

The assumption that u is bounded is key; indeed, the function $u(x) := \log x$ is harmonic in $D := \{x \in \mathbb{R}^2\colon |x| > 1\}$ yet it vanishes on ∂D and so (17.7) cannot hold. This is because the "boundary at infinity" carries a non-trivial contribution under the $n \to \infty$ limit of (17.11).

The assumption (17.6) is valid for all D bounded by the fact that the d-dimensional Brownian motion is unbounded almost surely. It is worth noting that the representation (17.7) also implies a uniqueness clause. Indeed, we have:

Corollary 17.3 (Maximum Principle) *Assuming* (17.6), *any bounded* $u \in C^2(D) \cap C(\overline{D})$ *satisfying* $\Delta u = 0$ *on* D *obeys the Maximum Principle,*

$$\sup_{x\in D} \bigl|u(x)\bigr| \leqslant \sup_{x \in \partial D} \bigl|u(x)\bigr| \tag{17.14}$$

In particular, under (17.6) *the Dirichlet problem in Definition 17.1 has a unique bounded solution, if a solution exists at all.*

Proof. The inequality (17.14) follows from (17.7). The difference of two solutions to (17.2–17.3) is a function $u \in C^2(D) \cap C(\overline{D})$ with $u = 0$ on ∂D. The Maximum Principle then gives $u = 0$ on $\overline{D}$. □

17.2 Non-uniqueness of solutions and Strong Markov property.

It has been a subject of interesting development in mathematics of the 19th and 20th century to realize that, even for continuous and bounded boundary values, the Dirichlet problem may either fail to be uniquely solvable or fail to be solvable. The proof of Theorem 17.2 highlights two mechanisms for this failure:

(1) If $P^x(\tau_D < \infty) < 1$, then some Brownian paths keep wandering forever inside D and, if $u(B_{t\wedge\tau_D})$ remains non-trivial along these, they make non-trivial contribution to (the $n \to \infty$ limit of) the expectation in (17.11).
(2) Even with $\tau_D < \infty$ a.s., when the Brownian motion is started from a point $x \in D$ very near some $x_0 \in \partial D$, there is no reason why B_{τ_D} should be near x_0 and thus why u defined by (17.7) should be continuous at all boundary points.

We start by analyzing (1) in some detail. As it turns out, condition (17.6) is in fact necessary for uniqueness of a bounded solution.

Proposition 17.4 *Let $D \subseteq \mathbb{R}^d$ be non-empty open and let $g\colon \partial D \to \mathbb{R}$ be bounded and measurable. Then for each $a \in \mathbb{R}$ and $x \in \overline{D}$,*

$$u(x) := E^x\big(g(B_{\tau_D})1_{\{\tau_D<\infty\}}\big) + aP^x(\tau_D = \infty) \tag{17.15}$$

defines $u \in C^2(D)$ such that $\Delta u = 0$ on D and $u = g$ on ∂D. (No statement of continuity at the boundary is made here.)

Fromula (17.15) indeed gives a mechanism for non-uniqueness because, once $P^x(\tau_D = \infty) \neq 0$ at some $x \in D$, varying the parameter a yields different functions with $u = g$ on ∂D. A key tool in the proof of this lemma is the following theorem, which is of independent interest:

Theorem 17.5 (Strong Markov property of Brownian motion) *Let B be a d-dimensional standard Brownian motion started from $x \in \mathbb{R}^d$, $\{\mathcal{F}_t\}_{t\geqslant 0}$ a Brownian filtration and T a finite stopping time for $\{\mathcal{F}_t\}_{t\geqslant 0}$. Then*

$$\big\{B_{T+t} - B_T\colon t \geqslant 0\big\} \tag{17.16}$$

is a d-dimensional standard Brownian motion independent of $\mathcal{F}_{T+}$ under P^x.

Here and in the proof we will use the following concept:

Definition 17.6 *Let T be a stopping time for filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Then*

$$\mathcal{F}_{T+} := \Big\{A \in \mathcal{F}\colon \big(\forall t \geqslant 0\colon A \cap \{T \leqslant t\} \in \mathcal{F}_{t+}\big)\Big\} \tag{17.17}$$

where $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ defines the right-continuous version $\{\mathcal{F}_{t+}\}_{t\geqslant 0}$ of $\{\mathcal{F}_t\}_{t\geqslant 0}$.

We observe the following facts, whose easy proofs we refer to the reader:

Lemma 17.7 *Let $\{\mathcal{F}_t\}_{n\geqslant 1}$ be a filtration and T, S and $\{T_n\}_{n\geqslant 1}$ stopping times. Then:*

(1) $\mathcal{F}_{T+} = \bigl\{A\in\mathcal{F}\colon (\forall t\geqslant 0\colon A\cap\{T<t\}\in\mathcal{F}_t)\bigr\}$
(2) $S\leqslant T \;\Rightarrow\; \mathcal{F}_{S+}\subseteq\mathcal{F}_{T+}$ *and*

$$S<T\quad\text{on }\{T<\infty\}\quad\Rightarrow\quad\mathcal{F}_{S+}\subseteq\mathcal{F}_T \tag{17.18}$$

(3) $T_n\downarrow T \;\Rightarrow\; \mathcal{F}_{T+}=\bigcap_{n\geqslant 1}\mathcal{F}_{T_n+}$

Proof of Theorem 17.5. Assume first that T is a bounded stopping time. Pick $\lambda\in\mathbb{R}^d$ and note that, by Itô formula and the fact that $\langle\lambda\cdot B\rangle_t=\|\lambda\|_2^2 t$,

$$Z_t=\exp\Bigl\{\mathrm{i}\lambda\cdot B_t+\frac{\|\lambda\|_2^2}{2}t\Bigr\} \tag{17.19}$$

is a continuous local martingale and, since $\sup_{s\leqslant t}|Z_s|$ is bounded for each $t\geqslant 0$, a martingale under P^x. For T a bounded stopping time and $u\geqslant 0$, the fact that $T+u$ is a stopping time shows, via Optional Sampling Theorem, that

$$E^x\Bigl(\mathrm{e}^{\mathrm{i}\lambda\cdot B_{T+u}+\frac{\|\lambda\|_2^2}{2}(T+u)}\Bigm|\mathcal{F}_T\Bigr)=E^x(Z_{T+u}|\mathcal{F}_T)=Z_T=\mathrm{e}^{\mathrm{i}\lambda\cdot B_T+\frac{\|\lambda\|_2^2}{2}T}\quad\text{a.s.} \tag{17.20}$$

Using the $\mathcal{F}_T$-measurability and boundedness of the right-hand side, we recast this as

$$E^x\Bigl(\mathrm{e}^{\mathrm{i}\lambda\cdot(B_{T+u}-B_T)}\Bigm|\mathcal{F}_T\Bigr)=\mathrm{e}^{-\frac{\|\lambda\|_2^2}{2}u}\quad\text{a.s.} \tag{17.21}$$

Invoking this iteratively (similarly as, e.g., in (14.27)) we then derive

$$E^x\biggl(\exp\Bigl\{\mathrm{i}\sum_{k=1}^n\lambda_k\cdot(B_{T+t_k}-B_{T+t_{k-1}})\Bigr\}1_A\biggr)=\exp\Bigl\{-\frac12\sum_{k=1}^n\|\lambda_k\|_2^2(t_k-t_{k-1})\Bigr\}P^x(A) \tag{17.22}$$

for all $0\leqslant t_0<t_1<\dots<t_n$, all $\lambda_1,\dots,\lambda\in\mathbb{R}^d$ and all $A\in\mathcal{F}_{T+t_0}$.

At this point we have (17.22) for all bounded stopping times T. For T just finite we employ a limit argument. First, restrict to $t_0>0$ and take $T\wedge n$ instead of T. Noting that, with the help of Lemma 17.7,

$$\forall A\in\mathcal{F}_{T+t_0}\,\forall n\geqslant 1\colon\quad A\cap\{T\leqslant n\}\in\mathcal{F}_{T+t_0}\cap\mathcal{F}_n=\mathcal{F}_{(T+t_0)\wedge n}\subseteq\mathcal{F}_{T\wedge n+t_0} \tag{17.23}$$

the identity (17.22) applies for A replaced by $A\cap\{T\leqslant n\}$ and $B_{T+\cdot}$ by $B_{T\wedge n+\cdot}$, for all $A\in\mathcal{F}_{T+t_0}$. Passing to $n\to\infty$ inside the expectation using $1_{A\cap\{T\leqslant n\}}\to 1_A$, the continuity of Brownian motion and $A\mapsto P^x(A)$ and the Bounded Convergence Theorem on the left-hand side yield (17.22) for all $A\in\mathcal{F}_{T+t_0}$. Restricting to $A\in\mathcal{F}_{T+}$ with the help fo Lemma 17.7 we can take $t_0\downarrow 0$ using the continuity of Brownian motion and, one more time, the Bounded Convergence Theorem. This extends (17.22) to all finite stopping times T, all $t_0\geqslant 0$ and all $A\in\mathcal{F}_{T+}$.

Invoking the Cramér-Wold device, from (17.22) — note that $\|\cdot\|_2$ is the Euclidean norm — we get that the increments

$$\bigl\{B_{T+t_i}-B_{T+t_{i-1}}\colon i=1,\dots,n\bigr\} \tag{17.24}$$

are independent, and independent of $\mathcal{F}_{T+}$, with

$$B_{T+t_i} - B_{T+t_{i-1}} = \mathcal{N}(0, (t_i - t_{i-1})1) \tag{17.25}$$

for 1 the unit $d \times d$-matrix. As $t \mapsto B_{T+t} - B_T$ is continuous, $B_{T+\cdot} - B_T$ is a standard Brownian motion independent of $\mathcal{F}_{T+}$. □

Note that Theorem 17.5 generalizes the Brownian shift symmetry (see Proposition 5.1) to shifts by a stopping time. As the shift can be thought of a statement of the *Markov property*, the shift by a stopping time is referred to as the Strong Markov property. What this term amounts to expressed in:

Corollary 17.8 *Under the setting and notation of Theorem 17.5, conditional on $\mathcal{F}_{T+}$ the process $\{B_{T+t} \colon t \geqslant 0\}$ has the law of the standard Brownian motion started at B_T.*

The result has another useful consequence:

Corollary 17.9 (Blumenthal's zero-one law) *Let B denote the d-dimensional standard Brownian motion and let $\mathcal{F}_t^B := \sigma(B_s \colon s \leqslant t)$. Then for all $x \in \mathbb{R}^d$,*

$$\forall A \in \mathcal{F}_{0+}^B \colon \quad P^x(A) \in \{0, 1\} \tag{17.26}$$

Proof. As $\{\mathcal{F}_t^B\}_{t \geqslant 0}$ is a Brownian filtration, Theorem 17.5 for $T := 0$ tells us that $\mathcal{F}_{0+}^B$ is independent of $\sigma(B_t \colon t \geqslant 0)$ and, in particular, of $\mathcal{F}_{0+}^B$. Hence, $\forall A, A' \in \mathcal{F}_{0+}^B$ we have $P^x(A \cap A') = P^x(A)P^x(A')$. Taking $A' = A$ yields the claim. □

We remark that Blumenthal's zero-one law explains why the *limes superior* in Khinchin's Law of the Iterated Logarthim takes an a.s. constant value. Returning to the problem of non-uniqueness of solutions to the Dirichlet problem, we now give:

Proof of Proposition 17.4. Pick $x \in D$ and start with $u(x) = E^x(g(B_{\tau_D})1_{\{\tau_D < \infty\}})$. Let $r > 0$ be so small that the Euclidean ball $B(x,r)$ obeys $\overline{B(x,r)} \subseteq D$. Abbreviate $T := \tau_{B(x,r)}$ and denote $W_t := B_{T+t}$. Abbreviating $\tau_D^W := \inf\{t \geqslant 0 \colon W_t \notin D\}$, on $\{\tau_D < \infty\}$ (which forces $T < \infty$) we have

$$\tau_D = T + \tau_D^W \quad \wedge \quad B_{\tau_D} = W_{\tau_D^W} \tag{17.27}$$

and thus, by a plain rewrite of the quantities under conditional expectation,

$$E^x\Big(g(B_{\tau_D})1_{\{\tau_D < \infty\}}\Big|\,\mathcal{F}_T\Big) = E^x\Big(g(B_{\tau_D^W})1_{\{\tau_D^W < \infty\}}\Big|\,\mathcal{F}_T\Big) \quad \text{on } \{T < \infty\} \tag{17.28}$$

By Corollary 17.8, conditional on $\mathcal{F}_T$, we get $W_t = B_T + \widetilde{B}_t$, where $\widetilde{B}$ is a standard Brownian motion (started at 0) independent of $\mathcal{F}_T$. Using the definition of u, the expectation on the right of (17.28) thus equals $u(B_T)$ P^x-a.s. As $P^x(\tau_{B(x,r)} < \infty) = 1$, we get

$$\forall x \in D\, \forall r > 0 \colon \quad \overline{B(x,r)} \subseteq D \quad \Rightarrow \quad u(x) = E^x\big(u(B_{\tau_{B(x,r)}})\big) \tag{17.29}$$

Multiplying by r^{d-1}, integrating with respect to the Lebesgue measure over an interval of r and using that the d-dimensional Brownian motion is rotationally invariant (see remarks after (5.8)) then shows that u obeys the Mean-Value Property (17.4). Hence we get $u \in C^2(D)$ and $\Delta u = 0$ on D.

The above takes care of the case $a = 0$; a completely analogous argument for $u(x) := P^x(\tau_D = \infty) = 1 - E^x(1 \cdot 1_{\{\tau_D < \infty\}})$ then deals with $a \neq 0$ as well. □

Having discussed situations with multiple solutions, the next topic is regularity at the boundary. We will do this (and more) in the next lecture.

Further reading: Karatzas-Shreve, Sections 4.1, 4.2AB