16. Bessel processes

We proceed to discuss some specific applications of the Itô formula and time change to Brownian motion. With the Lévy characterization in hand, we look back at the equation (13.46) satisfied by the radial process associated with the d-dimensional standard Brownian motion. For the diffusive term on the right we get:

Lemma 16.1 *Using the notation* (13.44), *the process*

$$\widetilde{B}_t := \sum_{i=1}^{d} \int_0^t \frac{1}{R_s} B_s^{(i)} \mathrm{d}B_s^{(i)} \tag{16.1}$$

is a standard Brownian motion.

Proof. Clearly, $\widetilde{B}$ takes the form (14.20) with (14.21) showing its quadratic variation process to be

$$\langle \widetilde{B} \rangle_t = \sum_{i=1}^{d} \int_0^t \frac{1}{R_s^2} [B_s^{(i)}]^2 \mathrm{d}s = \int_0^t \mathrm{d}s = t \tag{16.2}$$

Theorem 14.6 shows that $\widetilde{B}$ is a standard Brownian motion as claimed. □

We now put forward:

Definition 16.2 *Let $d \in \mathbb{R}$. A d-dimensional* Bessel process *is a continuous $[0,\infty)$-valued stochastic process (or positive semimartingale) $\{X_t \colon t \geqslant 0\}$ such that, denoting*

$$\tau_0 := \inf\{t \geqslant 0 \colon X_t = 0\} \tag{16.3}$$

we have $\forall t \geqslant \tau_0 \colon X_t = 0$ and

$$\mathrm{d}X_t = \frac{d-1}{2X_t}\mathrm{d}t + \mathrm{d}B_t \quad \text{on } \{\tau_0 > t\} \tag{16.4}$$

where B is a (one-dimensional) standard Brownian motion.

We remark that (16.4) means that

$$X_t = X_0 + \int_0^t 1_{\{\tau_0 > s\}} \frac{d-1}{2X_s} \mathrm{d}s + B_{\tau_0 \wedge t} \tag{16.5}$$

We will write P^x for the law of this process subject to the initial condition

$$P^x(X_0 = x) = 1 \tag{16.6}$$

The requirement that X remains constant after hitting 0 for the first time technically makes X a Bessel process with *absorbing boundary condition* at 0. We make this choice as it is convenient in what follows.

Lemma 16.1 combined with the calculation leading to (13.46) show that the radial process of the d-dimensional standard Brownian motion stopped upon first hit of zero is the d-dimensional Bessel process according to Definition 16.2. In particular, the latter process exists for all integer $d \geqslant 1$.

A first natural question is now whether a d-dimensional Bessel process exists also for d's that are not positive integers and/or are not positive. In addition, we may worry about the process being unique in a sense to be specified. We will answer this — it does and it is! — using the theory of stochastic differential equations. The next question is how to decide whether $\tau_0 < \infty$ or $\tau_0 = \infty$ for the process started from $X_0 = x > 0$. In order to do this, we first note:

Lemma 16.3 *For $d \in \mathbb{R}$, let X be a d-dimensional Bessel process and let $\phi_d(x)\colon (0,\infty) \to \mathbb{R}$ be defined by*

$$\phi_d(x) := \begin{cases} x^{2-d} & \text{if } d \neq 2 \\ \log x & \text{if } d = 2 \end{cases} \tag{16.7}$$

Then $\{\phi_d(X_t)\colon t \geqslant 0\}$ is a local martingale.

Proof. Let $\theta \in \mathbb{R}$. The Itô formula then shows

$$\begin{aligned} \mathrm{d}X_t^\theta &= \theta X_t^{\theta-1}\mathrm{d}X_t + \frac{1}{2}\theta(\theta-1)X_t^{\theta-2}\mathrm{d}\langle X\rangle_t \\ &= 1_{\{\tau_0>t\}}\theta X_t^{\theta-1}\mathrm{d}B_t + 1_{\{\tau_0>t\}}\Big[\theta\frac{d-1}{2} + \frac{1}{2}\theta(\theta-1)\Big]X_t^{\theta-2}\mathrm{d}t \end{aligned} \tag{16.8}$$

The coefficient of the drift term vanishes when $\theta = 2-d$ thus showing that $\{X_t^{2-d}\colon t \geqslant 0\}$ is a local martingale for all d. This yields a trivial process for $d = 2$, so in that case we instead look at

$$\begin{aligned} \mathrm{d}\log X_t &= \frac{1}{X_t}\mathrm{d}X_t - \frac{1}{2X_t^2}\mathrm{d}\langle X\rangle_t \\ &= 1_{\{\tau_0>t\}}\frac{1}{X_t}\mathrm{d}B_t + 1_{\{\tau_0>t\}}\Big(\frac{d-1}{2} + \frac{1}{2}\Big)\frac{1}{X_t^2}\mathrm{d}t \end{aligned} \tag{16.9}$$

and note that here the coefficient of the drift term vanishes exactly when $d = 2$. In light of the definition of ϕ_d, this proves the claim. □

With the help of the above local martingale, we now prove:

Lemma 16.4 *Let X be a d-dimensional Bessel process. Set, for each $a > 0$,*

$$\tau_a := \inf\big\{t \geqslant 0\colon X_t = a\big\} \tag{16.10}$$

Then

$$\forall 0 < a < x < b\colon \quad P^x(\tau_a \wedge \tau_b < \infty) = 1 \tag{16.11}$$

and

$$\forall 0 < a < x < b\colon \quad P^x(\tau_a < \tau_b) = \frac{\phi_d(b) - \phi_d(x)}{\phi_d(b) - \phi_d(a)} \tag{16.12}$$

Proof. Let X be given by (16.5) with $X_0 = x$ a.s. and assume that $\widetilde{B}$ is a standard Brownian motion independent of X and B. We assume that $\widetilde{B}$ is adapted to the filtration underlying our setting. Given $a \in (0,x)$ and $b \in (x,\infty)$, let Z be a continuous process such that, for each $t \geqslant 0$,

$$Z_t = \phi_d(X_0) + \int_0^t 1_{\{\tau_a\wedge\tau_b>s\}}\phi_d'(X_s)\mathrm{d}B_s + 1_{\{\tau_a\wedge\tau_b\leqslant t\}}(\widetilde{B}_t - \widetilde{B}_{\tau_a\wedge\tau_b}) \quad \text{a.s.} \tag{16.13}$$

Then

$$\forall t \leqslant \tau_a \wedge \tau_b\colon \quad Z_t = \phi_d(X_t) \tag{16.14}$$

In addition, letting

$$T(t) := \inf\{u \geqslant 0\colon \langle Z\rangle_u \geqslant t\} \tag{16.15}$$

and noting that

$$\langle Z\rangle_t = \int_0^t \big[1_{\{\tau_a\wedge\tau_b>s\}}\phi_d'(X_s)^2 + 1_{\{\tau_a\wedge\tau_b\leqslant s\}}\big]\mathrm{d}s, \tag{16.16}$$

the fact that $(\phi_d')^2$ is bounded and uniformly positive on $[a,b]$ shows that

$$\exists c_1, c_2 \in (0,\infty)\,\forall t \geqslant 0\colon \quad c_1 t \leqslant T(t) \leqslant c_2 t \tag{16.17}$$

By Theorem 15.1 $\{Z_{T(t)}\colon t \geqslant 0\}$ is the standard Brownian motion started at $\phi_d(x)$ and the event $\tau_a \wedge \tau_b = \infty$ corresponds to this Brownian motion staying between $\phi_d(a)$ and $\phi_d(b)$ for all times. This is readily checked to be a null event thus proving (16.11).

In order to prove (16.12) we proceed by a familiar argument. Let

$$M_t := \phi_d(X_{t\wedge\tau_a\wedge\tau_b}) \tag{16.18}$$

This process is a local martingale by Lemma 16.3 and, with all of its path bounded between $\phi_d(a)$ and $\phi_d(b)$, it is a martingale by Lemma 14.7. The Optional Stopping Theorem (namely, Theorem 14.6(2)) then shows

$$\begin{aligned}\phi_d(x) = E^x(M_0) &= E^x(M_{\tau_a\wedge\tau_b})\\ &= \phi_d(a)P^x(\tau_a<\tau_b) + \phi_d(x)P^x(\tau_b<\tau_a)\end{aligned} \tag{16.19}$$

Using that

$$P^x(\tau_b < \tau_a) = 1 - P^x(\tau_a < \tau_b) \tag{16.20}$$

by (16.11), the formula (16.12) now follows by a simple calculation. □

We are now ready to draw the main conclusions about Bessel processes:

Theorem 16.5 *Let $d \in \mathbb{R}$ and let $\{X_t\colon t \geqslant 0\}$ be a d-dimensional Bessel process started from $x > 0$. Then we have:*

(1) *if $d > 2$, then*

$$\tau_0 = \infty \;\wedge\; \inf_{t\geqslant 0} X_t > 0 \;\wedge\; \limsup_{t\to\infty} X_t = \infty \quad \text{a.s.} \tag{16.21}$$

(2) *if $d < 2$, then*

$$\tau_0 < \infty \;\wedge\; \sup_{t\geqslant 0} X_t < \infty \quad \text{a.s.} \tag{16.22}$$

(3) *if $d = 2$, then*

$$\tau_0 = \infty \;\wedge\; \forall a > 0\colon\, \tau_a < \infty \quad \text{a.s.} \tag{16.23}$$

and thus

$$\liminf_{t\to\infty} X_t = 0 \;\wedge\; \limsup_{t\to\infty} X_t = \infty \quad \text{a.s.} \tag{16.24}$$

Proof. We will rely heavily on (16.11–16.12) along with the fact that, regardless of d,

$$\tau_b \underset{b\to\infty}{\longrightarrow} \infty \tag{16.25}$$

which follows from the fact that X has continuous and thus locally bounded sample paths. We now proceed to deal separately with the cases $d > 2$, $d < 2$ and $d = 2$. Throughout we will only consider $a, b, x \in (0,\infty)$ subject to $0 < a < x < b < \infty$.

Assume first $d > 2$. Then $\phi_d(b) \to 0$ as $b \to \infty$. Using (16.25) and continuity of measure we then get

$$P^x(\tau_a < \infty) = \lim_{b\to\infty} P^x(\tau_a < \tau_b) = \frac{\phi_d(x)}{\phi_d(a)} \tag{16.26}$$

As $\phi_d(a) \to \infty$ as $a \downarrow 0$, this implies

$$\lim_{a\downarrow 0} P^x(\tau_a < \infty) = 0 \tag{16.27}$$

Picking a subsequence $\{a_n\}_{n\geqslant 1}$ with $a_n \downarrow 0$ such that $P^x(\tau_{a_n} < \infty) \leqslant 2^{-n}$, the Borel-Cantelli lemma gives $P(\tau_{a_n} < \infty \text{ i.o}(n)) = 0$ proving $\inf_{t\geqslant 0} X_t > 0$ a.s. In particular, $\tau_0 = \infty$ a.s. and, since $\tau_a = \infty$ for $a < \inf_{t\geqslant 0} X_t$, also that $\tau_b < \infty$ a.s. for all $b > x$, thanks to (16.11). This proves (16.21).

Next let us consider the cases $d < 2$. Then $\phi_d(a) \to 0$ as $a \downarrow 0$ and, since

$$\tau_0 = \lim_{a\downarrow 0} \tau_a \tag{16.28}$$

by continuity of X, we have

$$P^x(\tau_0 < \tau_b) = \lim_{a\downarrow 0} P^x(\tau_a < \tau_b) = 1 - \frac{\phi_d(x)}{\phi_d(b)} \tag{16.29}$$

Taking $b \to \infty$ while noting that, in this case, $\phi_d(b) \to \infty$ as $b \to \infty$ shows

$$P^x(\tau_0 < \infty) = 1 \tag{16.30}$$

But then we must have $\sup_{t\geqslant 0} X_t < \infty$ by continuity of X, proving (16.22).

Turning to the case $d = 2$, here we note that here

$$P^x(\tau_a < \tau_b) = \frac{\log b - \log x}{\log b - \log a} \tag{16.31}$$

The observation (16.25) then gives

$$P^x(\tau_a < \infty) = \lim_{b\to\infty} P^x(\tau_a < \tau_b) = 1 \tag{16.32}$$

and thus also

$$P^x(\tau_b = \infty) \leqslant \lim_{a\downarrow 0} P^x(\tau_a < \tau_b) = 0 \tag{16.33}$$

It follows that $\tau_c < \infty$ P^x-a.s. for each $c \in (0,\infty)$ regardless of its relation to x. In particular, X is unbounded a.s. which forces $\tau_0 = \infty$ a.s. $\square$

We note that the *limes superior* in (16.21) can be replaced by actual limit. This is not the case in (16.24).

Thanks to fact that the radial process of d-dimensional standard Brownian motion is a d-dimensional Bessel process, Theorem 16.5 now gives:

Corollary 16.6 (Recurrence/transience of Brownian motion) *For d-dimensional standard Brownian motion B, the following holds:*

(1) *in $d = 1$, B is recurrent to points, meaning that*

$$\{B_t : t \geqslant 0\} = \mathbb{R} \quad \text{a.s.} \tag{16.34}$$

(2) *in $d = 2$, B is not recurrent to points but is recurrent to open balls, meaning that*

$$\forall x \in \mathbb{R}^2 \smallsetminus \{0\}: \quad P(\exists t \geqslant 0 : B_t = x) = 0 \tag{16.35}$$

yet

$$\{B_t : t \geqslant 0\} \text{ is dense in } \mathbb{R}^2 \quad \text{a.s.} \tag{16.36}$$

(3) *in $d \geqslant 3$, B is not even recurrent to open balls, meaning that*

$$\forall x \in \mathbb{R}^3 \smallsetminus \{0\}: \quad \inf_{t \geqslant 0} |B_t - x| > 0 \quad \text{a.s.} \tag{16.37}$$

Notice that the case $d = 2$ is formally distinct from that of two-dimensional simple random walk, which is recurrent to points even in $d = 2$. The conclusion (16.35) also implies that

$$\lambda\Big(\{B_t : t \geqslant 0\}\Big) = 0 \quad \text{a.s.} \tag{16.38}$$

by a simple Fubini-Tonelli based argument relying on joint measurability of B. The Brownian path is thus dense in $\mathbb{R}^2$ yet of vanishing Lebesgue measure.

For the Bessel process, we in turn get:

Corollary 16.7 *For any d-dimensional Bessel process started from $x > 0$,*

$$d \geqslant 2 \quad \Rightarrow \quad \tau_0 = \infty \quad \text{a.s.} \tag{16.39}$$

and

$$d < 2 \quad \Rightarrow \quad \tau_0 < \infty \quad \text{a.s.} \tag{16.40}$$

An obvious technical advantage of the cases with $d \geqslant 2$ is that the stochastic differential equation (16.4) applies to all times. The cases $d < 2$ will serve as an example of a solution to a stochastic differential equation that hits a singularity in finite time a.s.

We leave it to the reader to check that Corollaries 16.6–16.7 indeed follow from the conclusions of Theorem 16.5.

Further reading: Karatzas-Shreve, Section 3.3C