15. Time change to Brownian motion

In this lecture we push Theorem 14.6 further by effectively removing the assumption that the quadratic variation process at time t equals t. This requires performing a *time change* which will become one of our standard tools in solving stochastic differential equations. The principal result is as follows:

Theorem 15.1 (Time change to Brownian motion) *Let X be a continuous process such that, for some $Y \in \mathcal{V}^{\text{loc}}$ and each $t \geqslant 0$,*

$$X_t = \int_0^t Y_s \mathrm{d}B_s \tag{15.1}$$

Assume that

$$\int_0^\infty Y_s^2 \mathrm{d}s = \infty \tag{15.2}$$

and, for each $t \geqslant 0$, set

$$T(t) := \inf\Big\{u \geqslant 0 \colon \int_0^u Y_s^2 \mathrm{d}s \geqslant t\Big\} \tag{15.3}$$

Then $T(t)$ is a stopping time with $T(t) \to \infty$ as $t \to \infty$. Moreover, $\{X_{T(t)} \colon t \geqslant 0\}$ is a standard Brownian motion, modulo a change on a null set.

Observe that (15.2) ensures that $T(t)$ is finite for all $t \geqslant 0$. Thanks to the continuity of $u \mapsto \int_0^u Y_s^2 \mathrm{d}s$ implied by $Y \in \mathcal{V}^{\text{loc}}$, we thus have

$$\forall t \geqslant 0\colon \qquad \int_0^{T(t)} Y_s^2 \mathrm{d}s = t \tag{15.4}$$

We remark that one could as well define $T(t)$ for $t > 0$ by

$$\sup\Big\{u \geqslant 0 \colon \int_0^u Y_s^2 \mathrm{d}s < t\Big\} \tag{15.5}$$

which (for $t > 0$) yields the same quantity as in (15.3) thanks to (15.4). The formula (15.3) is better suited because it also gives $T(0) = 0$ which for the definition via (15.5) had to be assumed separately. As noted as well, writing "$\leqslant$" instead of "$<$" in (15.5) (or *vice versa* in (15.3)) would lead to an optional time which we could still work with but only assuming that the underlying filtration is right continuous.

15.1 Continuous time martingales and stopping times.

The proof of Theorem 15.1 is based on the theory of continuous martingales. We will thus start reviewing the necessary facts emphasizing, particularly, the aspects arising from the continuum nature of the index set. The reader should perhaps start by re-reading the definition of a continuous martingale (Definition 9.1) and a stopping time (Definition 10.2). Next we put forward:

Definition 15.2 *Let T be a stopping time for filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$. An event in*

$$\mathcal{F}_T := \Big\{A \in \mathcal{F} \colon \big(\forall t \geqslant 0 \colon A \cap \{T \leqslant t\} \in \mathcal{F}_t\big)\Big\} \tag{15.6}$$

is then said to be measurable by the stopping time T.

In order to avoid confusion, we point out that $\mathcal{F}_T$ is not to be read as $\mathcal{F}_t|_{t=T}$ — which is a random element of the filtration. Rather, $\mathcal{F}_T$ is a collection of sets determined by the random variable T. We now observe:

Lemma 15.3 *For any stopping times T and S of filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$,*

(1) *$\mathcal{F}_T$ (and $\mathcal{F}_S$) is a σ-algebra and T is $\mathcal{F}_T$-measurable.,*
(2) *$\mathcal{F}_T \cap \mathcal{F}_S = \mathcal{F}_{T\wedge S}$ and so, in particular,*

$$S \leqslant T \quad \Rightarrow \quad \mathcal{F}_S \subseteq \mathcal{F}_T \tag{15.7}$$

Proof. (1) We readily check that $\mathcal{F}_T$ is closed under countable unions. That $\Omega \in \mathcal{F}_T$ and $\mathcal{F}_T$ is closed under complements follows from T being a stopping time. Hence $\mathcal{F}_T$ is a σ-algebra. From

$$\{T \leqslant u\} \cap \{T \leqslant t\} = \{T \leqslant u \wedge t\} \in \mathcal{F}_{u\wedge t} \subseteq \mathcal{F}_t \tag{15.8}$$

we get $\{T \leqslant u\} \in \mathcal{F}_T$ for each $u \geqslant 0$. As closed half-infinite intervals generate all Borel sets, T is $\mathcal{F}_T$-measurable. We leave the proof of (2) to a homework exercise. □

Our next point is to look at X evaluated at the stopping time T. This amounts to the consideration of the function $(X_T)(\omega) := X_{T(\omega)}(\omega)$ which makes sense (unless we have an interpretation of X_∞) provided T is finite. The question we are interested in is under what conditions besides X being adapted to a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is X_T an $\mathcal{F}_T$-measurable random variable. A sufficient condition for this is the subject of:

Lemma 15.4 *Let T be a finite stopping time for a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and X a process that is adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$. If X has left-continuous paths, then X_T is $\mathcal{F}_T$-measurable.*

We leave the proof of this lemma to homework while noting that the statement holds also for X with right continuous paths. If we wish to go beyond the one-sided continuous cases, a suitable notion of regularity of X to assume is the following:

Definition 15.5 *We say that ($\mathbb{R}$-valued) family $X = \{X_t \colon t \geqslant 0\}$ is* progressively measurable *if, for each $t \geqslant 0$ and each $A \in \mathcal{B}(\mathbb{R})$,*

$$\big\{(s,\omega) \in [0,t] \times \Omega \colon X_s(\omega) \in A\big\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t \tag{15.9}$$

As can be checked, a progressively measurable process is automatically jointly measurable but the converse is not true. One way to prove Lemma 15.4 is by first proving that processes with left continuous paths are progressively measurable:

Lemma 15.6 *Suppose X is progressively measurable. Then X_T is $\mathcal{F}_T$-measurable for each finite stopping time T.*

Proof. First we note that, for each $t \geqslant 0$ and each $A \in \mathcal{B}(\mathbb{R})$,

$$X_T^{-1}(A) \cap \{T \leqslant t\} = X_{T\wedge t}^{-1}(A) \cap \{T \leqslant t\} \tag{15.10}$$

and so to show that this set is in $\mathcal{F}_t$ it suffices to show that $X_{T\wedge t}$ is $\mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable. For this we write $X_{T\wedge t} = f \circ g$ where $f\colon [0,t]\times\Omega \to \mathbb{R}$ is the $\mathcal{B}([0,t])\otimes\mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable map $f(s,\omega) := X_s(\omega)$ — this is where progressive measurability enters — while $g\colon \Omega \to [0,t]\times\Omega$ is the $\mathcal{F}_t/\mathcal{B}([0,t])\otimes\mathcal{F}_t$-measurable map $g(\omega) := (T(\omega)\wedge s,\omega)$ — which by looking at product events reduces to the fact that $T\wedge t$ is $\mathcal{F}_t$-measurable, as follows from, e.g., Lemma 15.3. $\square$

We note that progressive measurability is apparently weaker than just joint measurability although, thanks to a theorem of K.L. Chung and J. Doob from 1965, an adapted, jointly measurable process admits a progressively measurable version. See Section 1.1 of Karatzas and Shreve for further discussion of this.

15.2 Optional Stopping/Sampling Theorem.

Returning to our main line of our discussion, the key result we need for the proof of Theorem 15.1 is then:

Theorem 15.7 (Optional Stopping/Sampling Theorem) *Let $M = \{M_t\colon t \geqslant 0\}$ be a continuous martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and let T and S be stopping times for $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that $S \leqslant T$. If*

(1) *either T is bounded, meaning $T \in L^\infty$, or*
(2) *$T < \infty$ a.s. and $\{M_t\colon t \geqslant 0\}$ is uniformly integrable,*

then $M_T \in L^1$ and

$$E(M_T|\mathcal{F}_S) = M_S \quad \text{a.s.} \tag{15.11}$$

In particular, $EM_T = EM_S = EM_0$.

Before we get to the proof note that the purpose of the Optional Stoping Theorem is to extend the equality $E(M_t) = E(M_0)$ from fixed (deterministic) t to a stopping time T. This requires certain conditions that, generally, trade assumptions on the stopping times against those on the martingale. Conditions (1) and (2) above match those used in discrete time setting; the other condition invoked in discrete time setting — namely, the stopping time has finite moment and the increments of the martingale are bounded — has no direct counterpart because being uniformly Lipschitz forces the martingale to be constant by Lemma 12.8. A version adapted to quadratic variation nonetheless exists; see the upcoming homework assignment.

The qualifier "sampling" in the title of Theorem 15.7 refers to the extension of conditional expectation version $\forall s \leqslant t\colon\ E(M_t|\mathcal{F}_s) = M_s$ a.s. of the martingale property to random (stopping) times $S \leqslant T$.

Proof of Theorem 15.7. The strategy of the proof in both cases is to first assume that T is bounded, prove the result for T and S discrete valued, then take limits to extend to the case of continuum-valued S and T (still with T bounded) and, to get also (2), remove the boundedness assumption by another limit.

Let us start by treating the cases when

$$\exists K \in (0,\infty)\colon\ S \leqslant T \leqslant K \tag{15.12}$$

Assuming first that S,T takes values in a finite set $\Sigma \subseteq [0,K]$, note that

$$|M_T| \leqslant \sum_{t\in\Sigma} |M_t| \tag{15.13}$$

implies $M_T \in L^1$, while the fact that $\{T = t\} = \{T \leqslant t\} \smallsetminus \bigcup_{t'\in\Sigma\cap[0,t)}\{T \leqslant t'\}$ gives

$$\forall A \in \mathcal{F}_T\ \forall t \in \Sigma\colon \quad A \cap \{T = t\} \in \mathcal{F}_t \ \wedge\ \{T = t\} \in \mathcal{F}_t \tag{15.14}$$

Using that $M_t = E(M_K|\mathcal{F}_t)$ a.s. for each $t \in \Sigma$, for any $A \in \mathcal{F}_T$ we then get

$$E(M_T 1_A) = \sum_{t\in\Sigma} E(M_t 1_{A\cap\{T=t\}}) = \sum_{t\in\Sigma} E(M_K 1_{A\cap\{T=t\}}) = E(M_K 1_A). \tag{15.15}$$

As M_T is $\mathcal{F}_T$-measurable (the full power of Lemma 15.4 is not required for this) hereby we conclude

$$M_T = E(M_K|\mathcal{F}_T) \text{ a.s.} \quad \text{and, similarly,} \quad M_S = E(M_K|\mathcal{F}_S) \text{ a.s.} \tag{15.16}$$

For $S \leqslant T$ we have $\mathcal{F}_S \subseteq \mathcal{F}_T$ by (15.7) and so (15.11) follows by the "smaller always wins" principle for conditional expectation.

Assume now that S,T are general stopping times still subject to (15.12). Define, for each integer $N \geqslant 1$, their approximations

$$T_N := 2^{-N}\lceil 2^N T\rceil \ \wedge\ S_N := 2^{-N}\lceil 2^N S\rceil \tag{15.17}$$

Then T_N and S_N are discrete-valued stopping times with $S_N \leqslant T_N \leqslant K+1$ and so, by the previous reasoning,

$$M_{T_N} = E(M_{K+1}|\mathcal{F}_{T_N}) \text{ a.s.} \ \wedge\ M_{S_N} = E(M_{K+1}|\mathcal{F}_{S_N}) \tag{15.18}$$

Let $A \in \mathcal{F}_S$. Then $S \leqslant S_N \leqslant T_N$ implies $A \in \mathcal{F}_{S_N} \subseteq \mathcal{F}_{T_N}$ and so the definition of conditional expectation shows

$$E(M_{T_N} 1_A) = E(M_{K+1} 1_A) = E(M_{S_N} 1_A) \tag{15.19}$$

Thanks to the continuity of M and the fact that $T_N \downarrow T$ and $S_N \downarrow S$, we have $M_{T_N} \to M_T$ and $M_{S_N} \to M_S$ pointwise. The formulas (15.19) in turn show that the families $\{M_{T_N}\colon N \geqslant 1\}$ and $\{M_{S_N}\colon N \geqslant 1\}$ are uniformly integrable. This proves $M_T, M_S \in L^1$ and permits passing the $N \to \infty$ limit inside the expectations in (15.19) to get

$$\forall A \in \mathcal{F}_S\colon \quad E(M_T 1_A) = E(M_S 1_A) \tag{15.20}$$

Since M_S is $\mathcal{F}_S$-measurable by Lemma 15.4, we have (15.11) assuming (15.12).

The above already proves (1). We leave the proof of (2), which also requires the Martingale Convergence Theorem to furnish the "terminal point" M_∞ of the martingale M, to a homework exercise. □

Note that the proof works fine even for just right-continuous martingales although, in that case, we need to supply a version of Lemma 15.4 for right-continuous processes to ensure $\mathcal{F}_S$-measurability of M_S. It is a convenient fact that, for martingales, the right-continuity is more or less automatic. Indeed, a.e. path of a martingale admits left and right limits at all points and (assuming $\mathcal{F}_0$ contains all P-null sets) the right-limit process remains a martingale with respect to the right-continuous version of the filtration. See Section 1.3A of Karatzas and Shreve.

15.3 Proof of time change to Brownian motion.

We are now ready to move to:

Proof of Theorem 15.1. Consider the process (15.1) and let $T(t)$ be as in (15.3). To see that $T(t)$ is a stopping time note that, by (15.4),

$$\big\{T(t) > u\big\} = \bigg\{\int_0^u Y_s^2 \mathrm{d}s < t\bigg\}. \tag{15.21}$$

As is checked with the help of the Monotone Class Theorem, the integral is $\mathcal{F}_u$-measurable and so $\{T(t) \leqslant u\} \in \mathcal{F}_u$ as desired. That $T(t) < \infty$ a.s. for all $t \geqslant 0$ follows from (15.2).

Next fix $\lambda \in \mathbb{R}$ and consider the process

$$M_u := \exp\Big\{\mathrm{i}\lambda X_{T(t)\wedge u} + \frac{\lambda^2}{2}\langle X\rangle_{T(t)\wedge u}\Big\} \tag{15.22}$$

Since

$$X_{T(t)\wedge u} = \int_0^u Y_s 1_{\{T(t)>s\}} \mathrm{d}B_s \ \wedge \ \langle X\rangle_{T(t)\wedge u} = \int_0^u Y_s^2 1_{\{T(t)>s\}} \mathrm{d}s \tag{15.23}$$

the Itô formula shows, once again, that M is a local martingale. Since $\langle X\rangle_{T(t)\wedge u} \leqslant t$ by definition of $T(t)$, we get that M is also bounded in the sense that $\sup_{u\geqslant 0}|M_u| \in L^\infty$ for each $t \geqslant 0$. Lemma 14.7 then shows that M is a martingale.

Since $T(t) < \infty$ a.s., the Optional Sampling Theorem (namely, Theorem 15.7(2) applied separately to the real and imaginary part of M) gives

$$\forall 0 \leqslant s < t\colon \quad E\big(M_{T(t)} \,\big|\, \mathcal{F}_{T(s)}\big) = M_{T(s)} \quad \text{a.s.} \tag{15.24}$$

But $\langle X\rangle_{T(t)} = t$ and so we write this as

$$\forall 0 \leqslant s < t\colon \quad E\Big(\mathrm{e}^{\mathrm{i}\lambda(X_{T(t)}-X_{T(s)})} \,\Big|\, \mathcal{F}_{T(s)}\Big) = \mathrm{e}^{-\frac{\lambda^2}{2}(t-s)} \quad \text{a.s.} \tag{15.25}$$

where we also used that $X_{T(s)}$ is $\mathcal{F}_{T(s)}$-measurable. Using the same argument in the proof of Theorem 14.6 we conclude that the process $\{X_{T(t)}\colon t \geqslant 0\}$ has independent increments with $X_{T(t)} - X_{T(s)} = \mathcal{N}(0, t-s)$.

In order to show that $X_{T(\cdot)}$ is a standard Brownian motion modulo a change on a null set, we need to prove that $t \mapsto X_{T(t)}$ is continuous a.s. For this we first note that, since $\{X_{T(t)}\colon t \geqslant 0\}$ has the same finite-dimensional distributions as the standard Brownian motion, we have

$$t \mapsto X_{T(t)} \text{ is locally uniformly continuous on } \mathbb{Q} \cap [0,\infty) \text{ a.s.} \tag{15.26}$$

To get continuity on $(0,\infty)$ it then suffices to observe that

$$t \mapsto T(t) \quad \text{is left continuous on } (0,\infty) \tag{15.27}$$

which is gleaned from the fact that the set in (15.21) is open. In order to get continuity at time $t = 0$ a.s. we note that, by the fact that the integral representation $X_{T(t)\wedge u} =$

$\int_0^u Y_s 1_{\{T(t)>u\}} \mathrm{d}B_s$ a.s. along with Itô isometry ensure that

$$X_{T(t)\wedge u} \in L^2 \quad \text{with} \quad E\big(X^2_{T(t)\wedge u}\big) = E\big(\langle X\rangle_{T(t)\wedge u}\big) \leqslant t \tag{15.28}$$

By above continuity argument and the Optional Sampling Theorem we conclude that $\{X_{T(t)\wedge u} \colon t > 0\}$ is a continuous L^2-martingale for the filtration $\{\mathcal{F}_{T(t)} \colon t > 0\}$. Doob's L^2-inequality (see Lemma 9.3) still applies (ignoring the value at $t = 0$) and shows, for each $t > 0$ and each $\lambda > 0$,

$$\begin{aligned} P\Big(\sup_{0<s\leqslant t} |X_{T(s)}| > \lambda\Big) &= \lim_{u\to\infty} P\Big(\sup_{0<s\leqslant t} |X_{T(s)\wedge u}| > \lambda\Big) \\ &\leqslant \limsup_{u\to\infty} \frac{1}{\lambda^2} E\big(X^2_{T(t)\wedge u}\big) \leqslant \frac{t}{\lambda^2}, \end{aligned} \tag{15.29}$$

where the first equality follows from the fact that $X_{T(s)\wedge u} = X_{T(s)}$ for all $s \in (0,t]$ on $\{T(t) \leqslant u\}$ and that $P(T(t) \leqslant u) \to 1$ as $u \to \infty$. Taking $t := 1/n^{-4}$ and $\lambda := 1/n$ and invoking the Borel-Cantelli lemma shows

$$X_{T(t)} \underset{t\downarrow 0}{\overset{\text{a.s.}}{\longrightarrow}} 0 \tag{15.30}$$

As $T(0) = 0$ and so $X_{T(0)} = 0$ by assumed continuity of X, we conclude that, away from the null sets in (15.26) and (15.30), $t \mapsto X_{T(t)}$ is continuous on $[0,\infty)$ and is thus a standard Brownian motion. □

We remark that the care we treat continuity in the above proof is necessary because $t \mapsto T(t)$ fails to be continuous whenever Y vanishes a.s. on a non-degenerate interval of times. While the above proof is cast for X given as a single stochastic integral, a completely analogous argument works for X taking the form (14.20). Indeed, all we need to do is to replace the integral in (15.3) by the sum in (14.21).

15.4 Extensions.

The setting of Theorem 15.1 contains two assumptions that are sometimes worthy of relaxing. The first one is that $Y \in \mathcal{V}^{\text{loc}}$ where we may not want to assume that $\int_0^t Y_s^2 \mathrm{d}s$ is always finite for all $t \geqslant 0$; meaning that

$$\tau := \inf\Big\{t \geqslant 0 \colon \int_0^t Y_s^2 \mathrm{d}s = \infty\Big\} \tag{15.31}$$

is finite with positive probability. In this case $T(t) \to \infty$ as $t \uparrow \tau$ and the above proof still works, except that only the portion of X corresponding to the interval of times for which the quadratic variation is finite is used.

Concerning the natural question what happens with $t \mapsto X_{T(t)}$ as $t \uparrow \tau$, modest localization arguments applied on top of the conclusion of Theorem 15.1 show:

Theorem 15.8 (Blow up in finite time) *Suppose Y is an adapted and jointly measurable process such that τ from* (15.31) *obeys $P(\tau > 0) = 1$ yet possibly $P(\tau < \infty) > 0$. Then the* *limit in* (10.4) *exists for all* $t \in [0, \tau)$ *and defines a continuous process* $\{X_t \colon t \in [0, \tau)\}$ *such that*

$$X_t = \int_0^t Y_s \mathrm{d}B_s \quad \text{a.s. on } \{\tau > t\} \tag{15.32}$$

The process X *does not extend continuously to* $t = \tau$ *when* $\tau < \infty$ *as we have*

$$\limsup_{t \uparrow \tau} X_t = +\infty \quad \wedge \quad \liminf_{t \uparrow \tau} X_t = -\infty \quad \text{a.s.} \tag{15.33}$$

Part (15.32) allows us to work with the stochastic integral up to the time when the associated quadratic variation process diverges. Part (15.32) in turn shows that this divergence is a fundamental obstruction as no "improper" integral can be defined at the point of divergence. We leave the proof of Theorem 15.8 to homework.

Another point where the setting of Theorem 15.8 may need to be relaxed is the assumption (15.2). This ensures that $T(t) < \infty$ for each $t \geqslant 0$ thus resulting in $X_{T(t)}$ being defined for all $t \geqslant 0$. In this case we get the following:

Theorem 15.9 (Time change with finite time horizon) *Consider a probability space supporting a Brownian motion* $B = \{B_t \colon t \geqslant 0\}$ *and a Brownian filtration* $\{\mathcal{F}_t\}_{t \geqslant 0}$. *Let* $\tau^\star$ *be a positive stopping time for* $\{\mathcal{F}_t\}_{t \geqslant 0}$ *and* $X = \{X_t \colon t < \tau^\star\}$ *a family of random variables such that* $t \mapsto X_t$ *is continuous on its domain and, for some* $Y \in \mathcal{V}^{\text{loc}}$ *and all* $t \geqslant 0$,

$$X_t = \int_0^t Y_s \mathrm{d}B_s \quad \text{a.s. on } \{\tau^\star > t\} \tag{15.34}$$

Assume the probability space supports another standard Brownian motion $\widetilde{B} = \{\widetilde{B}_t \colon t \geqslant 0\}$ *that is independent of* X, Y *and* B. *Consider the random variable*

$$T^\star := \int_0^{\tau^\star} Y_s^2 \, \mathrm{d}s \tag{15.35}$$

and, for $t \in [0, T^\star)$, *let* $T(t)$ *be defined by* (15.3). *Then*

$$X_\infty := \lim_{t \uparrow T^\star} X_{T(t)} \quad \text{exists a.s. on } \{T^\star < \infty\} \tag{15.36}$$

and the process $W = \{W_t \colon t \geqslant 0\}$ *defined by*

$$W_t := \begin{cases} X_{T(t)}, & \text{for } t < T^\star, \\ X_\infty + \widetilde{B}_t - \widetilde{B}_{T^\star}, & \text{for } t \geqslant T^\star, \end{cases} \tag{15.37}$$

is a standard Brownian motion, modulo a change on a null set.

We again leave this to a (potential) homework exercise as it requires, mostly, steps and tools that have been used above. We remark that all these results can be proved for general continuous local martingales, using the generalization of the Itô integral that replaces the standard Brownian motion by a continuous local martingale.

Further reading: Karatzas-Shreve, Section 3.4B