14. Product rule and Lévy's characterization of Brownian motion

We will now give two interesting consequences of the Itô formula.

14.1 Product rule and the Fisk-Stratonovich integral.

Noting that this formula is basically a statement of the chain rule, we now examine the validity of the product rule.

Theorem 14.1 (Product rule) *Let X and $\widetilde{X}$ be semimartingales (under the same Brownian filtration). Then so is $X\widetilde{X} = \{X_t\widetilde{X}_t : t \geqslant 0\}$ and*

$$\mathrm{d}(X\widetilde{X})_t = \widetilde{X}\,\mathrm{d}X_t + X_t\mathrm{d}\widetilde{X}_t + \mathrm{d}\langle X, \widetilde{X}\rangle_t \tag{14.1}$$

where $\{\langle X, \widetilde{X}\rangle_t : t \geqslant 0\}$ is the cross-variation process

$$\langle X, \widetilde{X}\rangle_t := \frac{1}{4}\langle X + \widetilde{X}\rangle_t - \frac{1}{4}\langle X - \widetilde{X}\rangle_t \tag{14.2}$$

Proof. Using the Itô formula in differential notation (in which you read just the top signs or the bottom signs)

$$\mathrm{d}(X_t \pm \widetilde{X}_t)^2 = 2\Big[X_t\,\mathrm{d}X_t + \widetilde{X}_t\,\mathrm{d}\widetilde{X}_t \pm (\widetilde{X}_t\,\mathrm{d}X_t + X_t\,\mathrm{d}\widetilde{X}_t)\Big] + \mathrm{d}\langle X \pm \widetilde{X}\rangle_t \tag{14.3}$$

The claim now follows from $4X_t\widetilde{X}_t = (X_t + \widetilde{X}_t)^2 - (X_t - \widetilde{X}_t)^2$. □

The cross-variation process is a natural extension of the quadratic variation process. Indeed, from (14.2) and homogeneity of $X \mapsto \langle X\rangle$ we get

$$\langle X\rangle_t = \langle X, X\rangle_t. \tag{14.4}$$

While we chose to define the cross variation directly via quadratic variation, its actual meaning is traced back to:

Lemma 14.2 *Let X and $\widetilde{X}$ be semimartingales. Then for any $t \geqslant 0$ and any sequence of partitions $\{\Pi_n\}_{n\geqslant 1}$ of $[0, t]$, where $\Pi_n = \{0 = t_0 < t_1 < \dots < t_m(n) = t\}$, if $\|\Pi_n\| \to 0$ then*

$$\sum_{i=1}^{m(n)} (\widetilde{X}_{t_i} - \widetilde{X}_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) \ \underset{n\to\infty}{\overset{P}{\longrightarrow}}\ \langle \widetilde{X}, X\rangle_t \tag{14.5}$$

Proof. Writing the left-hand side as $\frac{1}{4}[V_t^{(2)}(X + \widetilde{X}, \Pi_n) - V_t^{(2)}(X + \widetilde{X}, \Pi_n)]$, the claim follows from Theorem 12.5. □

The upshot of Theorem 14.1 is that, besides the chain rule failing for the Itô integral, the integration by parts formula fails as well. This is seen from

$$\int_0^t \widetilde{X}_s\,\mathrm{d}X_s = X_t\widetilde{X}_t - X_0\widetilde{X}_0 - \int_0^t X_s\,\mathrm{d}\widetilde{X}_s - \langle X, \widetilde{X}\rangle_t \tag{14.6}$$

This can be mended by coming up with a different version of the integral:

Definition 14.3 *Let X and $\widetilde{X}$ be semimartingales. Then*

$$\int_0^t \widetilde{X}_s \circ \mathrm{d}X_s := \int_0^t \widetilde{X}_s \,\mathrm{d}X_s + \frac{1}{2}\langle X, \widetilde{X}\rangle_t \tag{14.7}$$

is the Fisk-Stratonovich integral.

We then have:

Lemma 14.4 *Let X be a semimartingale. Then for all $t \geqslant 0$ and all $f \in C^1(\mathbb{R})$,*

$$\langle f \circ X, X\rangle_t = \int_0^t f'(X_t)\mathrm{d}\langle X\rangle_t \tag{14.8}$$

In particular, the chain rule

$$f(X_t) = f(X_0) + \int_0^t f(X_s) \circ \mathrm{d}X_s \tag{14.9}$$

holds for all $f \in C^2(\mathbb{R})$ and all $t \geqslant 0$. In addition, for any semimartingales X and $\widetilde{X}$ we also have the integration-by-parts formula

$$\int_0^t \widetilde{X}_s \circ \mathrm{d}X_s = X_t\widetilde{X}_t - X_0\widetilde{X}_0 - \int_0^t X_s \circ \mathrm{d}\widetilde{X}_s \tag{14.10}$$

We leave the short but instructive proof to homework. Thanks to (14.8), the integral (14.7) subsumes that introduced in (6.41). One can also wonder if there is an analogue of Lemma 6.8 which interprets the Stratonovich integral as the limit of Riemann sums under the midpoint rule. This comes in:

Lemma 14.5 *Let X and $\widetilde{X}$ be semimartingales. Then for any $t \geqslant 0$ and any sequence of partitions $\{\Pi_n\}$ of $[0,t]$, where $\Pi_n = \{0 = t_0 < t_1 < \dots < t_m(n) = t\}$, if $\|\Pi_n\| \to 0$ then*

$$\sum_{i=1}^{m(n)} \frac{\widetilde{X}_{t_i} + \widetilde{X}_{t_{i-1}}}{2}(X_{t_i} - X_{t_{i-1}}) \ \underset{n\to\infty}{\overset{P}{\longrightarrow}} \ \int_0^t \widetilde{X}_s \circ \mathrm{d}X_t \tag{14.11}$$

Proof. Subtracting half of the left-hand side of (14.5) from the quantity on the left of (14.11) yields the expression $\sum_{i=1}^{m(n)} \widetilde{X}_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})$. Our aim it to show that, similarly as in Lemma 11.8,

$$\sum_{i=1}^{m(n)} \widetilde{X}_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) \ \underset{n\to\infty}{\overset{P}{\longrightarrow}} \ \int_0^t \widetilde{X}_s \,\mathrm{d}X_t \tag{14.12}$$

Using a localization argument (whose details we leave to the reader), we may assume that $\widetilde{X}$ is bounded, say

$$\sup_{s\leqslant t} |\widetilde{X}_s| \leqslant K \tag{14.13}$$

for some deterministic $K > 0$.

We start by invoking Lemma 13.3 to write the difference of the Riemann sum and the integral on the right of (14.12) as

$$\sum_{i=1}^{m(n)} \int_{t_{i-1}}^{t_i} (\widetilde{X}_{t_{i-1}} - \widetilde{X}_s)\,\mathrm{d}X_s \tag{14.14}$$

For X of the form $\mathrm{d}X_t = U_t\mathrm{d}t + Y_t\mathrm{d}B_t$ this equals

$$\sum_{i=1}^{m(n)} \int_{t_{i-1}}^{t_i} (\widetilde{X}_{t_{i-1}} - \widetilde{X}_s)\,U_s\mathrm{d}s + \sum_{i=1}^{m(n)} \int_{t_{i-1}}^{t_i} (\widetilde{X}_{t_{i-1}} - \widetilde{X}_s)Y_s\,\mathrm{d}B_s \tag{14.15}$$

The first sum is bounded as

$$\Bigl|\sum_{i=1}^{m(n)} \int_{t_{i-1}}^{t_i} (\widetilde{X}_{t_{i-1}} - \widetilde{X}_s)\,U_s\mathrm{d}s\Bigr| \leqslant \mathrm{osc}_X\bigl([0,t],\|\Pi\|\bigr)\int_0^t |U_s|\mathrm{d}s \tag{14.16}$$

which tends to zero pointwise a.s. thanks to the assumption that the integral is finite a.s. The second sum in (14.15) can be written as

$$\sum_{i=1}^{m(n)} \int_{t_{i-1}}^{t_i} (\widetilde{X}_{t_{i-1}} - \widetilde{X}_s)Y_s\,\mathrm{d}B_s = \int_0^t \widetilde{Z}_s^{(n)}\,Y_s\,\mathrm{d}B_s \tag{14.17}$$

where

$$Z_s^{(n)} := \sum_{i=1}^{m(n)} (\widetilde{X}_{t_{i-1}} - \widetilde{X}_s)1_{[t_{i-1},t_i)}(s) \tag{14.18}$$

Thanks to the assumed continuity, $Z_s^{(n)} \to 0$ as $n \to \infty$. Using that and $\int_0^t Y_s^2\mathrm{d}s < \infty$ a.s. and that $Z^{(n)}$'s are uniformly bounded by (14.13), the Bounded Convergence Theorem gives $\int_0^t [Z_s^{(n)}\,Y_s]^2\mathrm{d}s \to 0$ a.s. as $n \to \infty$. Lemma 11.7 then implies

$$\int_0^t \widetilde{Z}_s^{(n)}\,Y_s\,\mathrm{d}B_s \ \underset{n\to\infty}{\overset{P}{\longrightarrow}}\ 0 \tag{14.19}$$

This proves (14.12); Lemma 14.2 and (14.7) then give (14.11). □

14.2 Lévy characterization of Brownian motion.

Another elegant application of the Itô formula is the characterization (proved by P. Lévy in 1949) of the standard Brownian motion as a continuous local martingale M with $\langle M\rangle_t = t$ for all $t \geqslant 0$. While the results works for all continuous local martingales, we only prove one for the local martingales arising from stochastic integrals.

Theorem 14.6 *Let $B^{(1)},\dots,B^{(d)}$ be independent standard Brownian motions and let X be a continuous process such that, for each $t \geqslant 0$,*

$$X_t = \sum_{k=1}^{d} \int_0^t Y_s^{(k)}\,\mathrm{d}B_s^{(k)} \tag{14.20}$$

for some $Y^{(1)},\dots,Y^{(d)}\in\mathcal{V}^{\text{loc}}$ *(defined via a common Brownian filtration). Then*

$$\forall t\geqslant 0\colon\ \langle X\rangle_t := \sum_{k=1}^d\int_0^t [Y_s^{(k)}]^2\,\mathrm{d}s = t \tag{14.21}$$

implies that X *is a standard Brownian motion.*

We note that, while the process $\{\langle X\rangle_t\colon t\geqslant 0\}$ can be defined intrinsically either by a limit of second variations or via the aforementioned Doob-Meyer decomposition, in (14.21) is simply designates the process that is naturally associated with the Itô terms arising for (14.20) in Theorem 13.5.

Proof of Theorem 14.6. Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be the Brownian filtration under which the integrals in (14.20) are defined. For any $\lambda\in\mathbb{R}$, let

$$Z_t := \mathrm{e}^{\mathrm{i}\lambda X_t+\frac{\lambda^2}{2}t} \tag{14.22}$$

Just as in (13.39), under (14.21) the Itô formula shows

$$\begin{aligned}\mathrm{d}Z_t &= \mathrm{i}\lambda Z_t\,\mathrm{d}X_t+\frac{\lambda^2}{2}Z_t\mathrm{d}t+\frac{1}{2}(\mathrm{i}\lambda)^2Z_t\mathrm{d}\langle X\rangle_t\\ &=\sum_{k=1}^d \mathrm{i}\lambda Z_tY_s^{(k)}\,\mathrm{d}B_s^{(k)}\end{aligned} \tag{14.23}$$

which means that Z is a local martingale. We now claim that Z is an actual martingale. We prove this in a bit more generality via:

Lemma 14.7 *Let* M *be a local martingale such that*

$$\forall t\geqslant 0\colon\qquad \sup_{s\leqslant t}|M_s|\in L^\infty \tag{14.24}$$

Then M *is a martingale.*

Postponing the proof of this lemma until we finish that of Theorem 14.6, we note that $|Z_t|\leqslant \mathrm{e}^{\frac{\lambda^2}{2}t}$ and so the conclusion of the lemma is available. The martingale property of Z then gives

$$\forall t\geqslant s\geqslant 0\colon\quad E\bigl(\mathrm{e}^{\mathrm{i}\lambda X_t+\frac{\lambda^2}{2}t}\,\big|\,\mathcal{F}_s\bigr)=\mathrm{e}^{\mathrm{i}\lambda X_s+\frac{\lambda^2}{2}s}\quad\text{a.s.} \tag{14.25}$$

which by the fact that X_s is $\mathcal{F}_s$-measurable transforms into

$$\forall t\geqslant s\geqslant 0\colon\quad E\bigl(\mathrm{e}^{\mathrm{i}\lambda (X_t-X_s)}\,\big|\,\mathcal{F}_s\bigr)=\mathrm{e}^{-\frac{\lambda^2}{2}(t-s)}\quad\text{a.s.} \tag{14.26}$$

Using this inductively we get, for any $0=t_0<t_1<\dots<t_n$ and any $\lambda_1,\dots,\lambda_n\in\mathbb{R}$,

$$E\biggl(\exp\Bigl\{\mathrm{i}\sum_{k=1}^n\lambda_k(X_{t_k}-X_{t_{k-1}})\Bigr\}\biggr)=\exp\Bigl\{-\frac{1}{2}\sum_{k=1}^n\lambda_k^2(t_k-t_{k-1})\Bigr\} \tag{14.27}$$

From the Cramér-Wold Theorem we then conclude that X has independent increments with $X_t-X_s=\mathcal{N}(0,t-s)$ for all $t\geqslant s\geqslant 0$. Since X has continuous paths, it is a standard Brownian motion. □

It remains to give:

Proof of Lemma 14.7. The local martingale property means that, for a non-decreasing sequence $\{T_n\}_{n\geqslant 1}$ of stopping times with $T_n \to \infty$ a.s. the process $\{M_{T_n\wedge t}\colon t \geqslant 0\}$ is a martingale for each $n \geqslant 1$. In particular, for each $t \geqslant s \geqslant 0$ we have

$$E(M_{T_n\wedge t}|\mathcal{F}_s) = M_{T_n\wedge s} \quad \text{a.s.} \tag{14.28}$$

Taking $n \to \infty$ we get $M_{T_n\wedge s} \to M_s$ a.s. so the issue is to take the same limit inside the conditional expectation. Here we use that (14.24) implies

$$E|M_{T_n\wedge t} - M_t| \underset{n\to\infty}{\longrightarrow} 0 \tag{14.29}$$

by the Bounded Convergence Theorem and so $M_{T_n\wedge t} \to M_t$ in L^1. The fact that the conditional expectation is an L^1-contraction then gives $E(M_{T_n\wedge t}|\mathcal{F}_s) \to E(M_t|\mathcal{F}_s)$ in L^1 and thus

$$\forall t \geqslant s \geqslant 0\colon \quad E(M_t|\mathcal{F}_s) = M_s \quad \text{a.s.} \tag{14.30}$$

thus proving that M is indeed a martingale. □

We remark that, while the boundedness assumption (14.24) is sufficient for the desired conclusion, it is definitely not required. On the other hand, as we will show a bit later, plain integrability of M_s, or even uniformly integrability of $\{M_s\colon s \leqslant t\}$, is not sufficient either. Indeed, in the absence of further information on the stopping times $\{T_n\}_{n\geqslant 1}$, (14.28) seems to need that $\{M_{T'}\colon T'$ stopping time with $T' \leqslant t\}$ is uniformly integrable.

In light of the remark after Theorem 13.5, the assumption that the Brownian motions in (14.20) are independent is actually not necessary. In fact, as noted above, the conclusion holds for all continuous local martingales.

Further reading: Sections 3.3AB of Karatzas-Shreve