13. Itô formula for semimartingales

We are now in a position to generalize the Itô formula for C^2-functions of standard Brownian motion to similar functions of semimartingales. For this we first extend the notion of the stochastic integrals to those with respect to semimartingales.

Definition 13.1 *Let $\{X_t\colon t \geqslant 0\}$ be a semimartingale of the form $\mathrm{d}X_t = U_t\mathrm{d}t + Y_t\mathrm{d}B_t$. For any $t \geqslant 0$ and any $\{Z_s\colon s \geqslant 0\}$ which is adapted, jointly measurable and obeys*

$$\int_0^t |Z_s||U_s|\mathrm{d}s < \infty \ \wedge \ \int_0^t |Z_s|^2 Y_s^2 \mathrm{d}s < \infty \quad \text{a.s.} \tag{13.1}$$

we then define

$$\int_0^t Z_s\mathrm{d}X_s := \int_0^t Z_s U_s \mathrm{d}s + \int_0^t Z_s Y_s \mathrm{d}B_s \tag{13.2}$$

Note that if Z has continuous paths, the function $s \mapsto Z_s$ is locally bounded and conditions (13.1) are thus ensured by those entering the designation of X as a semimartingale. Note also that, in light of Corollary 12.9, the representation of a semimartingale X is unique up to equivalence (12.30) and the integrals in (13.1) and those on the right of (13.2) are the same for any representatives of $\{U_s\colon s \geqslant 0\}$ and $\{Y_s\colon s \geqslant 0\}$. The integral $\int_0^t Z_s\mathrm{d}X_s$ is thus independent of the choice of the representatives as well.

We now claim:

Theorem 13.2 (Itô formula for semimartingales) *Let $\{X_t\colon t \geqslant 0\}$ be a semimartingale. Then for all $f \in C^2(\mathbb{R})$ also $\{f(X_t)\colon t \geqslant 0\}$ is a semimartingale and for all $t \geqslant 0$,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}\langle X\rangle_s \quad \text{a.s.} \tag{13.3}$$

where the first integral is in the sense of Definition 13.1 while the second integral is an ordinary Stieltjes integral. Both integrals exist and are finite a.s.

In order to make the statement completely explicit, assume that X has the differential form $\mathrm{d}X_t = U_t\mathrm{d}t + Y_t\mathrm{d}B_t$. Then (13.3) means that, for each $t \geqslant 0$, a.s.,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)U_s\mathrm{d}s + \int_0^t f'(X_s)Y_s\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(X_s)Y_s^2\mathrm{d}s \tag{13.4}$$

We will naturally write (13.3) in differential form as

$$\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)\mathrm{d}\langle X\rangle_t \tag{13.5}$$

For $X_t = B_t$, where B is the standard Brownian motion, we have $\langle B\rangle_t = t$ by Proposition 6.2 and so Theorem 13.2 subsumes Theorem 11.9.

Proof of Theorem 13.2. Assuming X takes the form $\mathrm{d}X_t = U_t\mathrm{d}t + Y_t\mathrm{d}B_t$ denote

$$\forall u \geqslant 0\colon \quad A_u := \int_0^u U_s\mathrm{d}s \ \wedge \ M_u := \int_0^u Y_s\mathrm{d}B_s \tag{13.6}$$

The proof follows closely the proof of Theorem 11.9 with two important differences. First, localization will be done first as it ensures a number of technical conditions that help in the rest of the proof. Second, the reduction of the quadratic terms $(X_{t_i} - X_{t_{i-1}})^2$ to $\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}}$ with their difference now controlled not via a fourth moment (which may not exist) but rather with the help of truncations induced in the localization step.

STEP 1 (localization): Given $K \geqslant 0$, consider the quantity

$$T_K := \inf\left\{ t \geqslant 0\colon \begin{array}{c} |X_t| \geqslant K \ \vee\ |f'(X_t)| \geqslant K \ \vee\ |f''(X_t)| \geqslant K \\ \vee\ |M_t| \geqslant K \ \vee\ \int_0^t |U_s| \mathrm{d}s \geqslant K \ \vee\ \int_0^t Y_s^2 \mathrm{d}s \geqslant K \end{array} \right\} \tag{13.7}$$

Since all t-dependent quantities are measurable, this is a stopping time. Moreover, by continuity of X, $f' \circ X$, $f'' \circ X$ and the assumption in Definition 12.1, $T_K \to \infty$ a.s. as $K \to \infty$. On $\{T_K > t\}$ we have $\forall s \in [0,t]\colon\ X_s = X_{T_K \wedge s}$ as well as

$$\int_0^t U_s \,\mathrm{d}s = \int_0^t U_s 1_{\{T_K > s\}} \mathrm{d}s \ \wedge\ \int_0^t Y_s \,\mathrm{d}B_s = \int_0^t Y_s 1_{\{T_K > s\}} \mathrm{d}B_s \tag{13.8}$$

and thus

$$X_{T_K \wedge s} = X_0 + \int_0^t U_s 1_{\{T_K > s\}} \mathrm{d}s + \int_0^t Y_s 1_{\{T_K > s\}} \mathrm{d}B_s \tag{13.9}$$

Working with the stopped process $\{X_{T_K \wedge s}\colon s \geqslant 0\}$ instead of $\{X_s\colon s \geqslant 0\}$, it thus suffices to prove the claim under the assumptions that, for all $t \geqslant 0$,

$$\sup_{s \leqslant t} |X_s| \leqslant K \ \wedge\ \sup_{s \leqslant t} |M_s| \leqslant K \ \wedge\ \sup_{s \leqslant t} \bigl|f'(X_s)\bigr| \leqslant K \ \wedge\ \sup_{s \leqslant t} \bigl|f''(X_s)\bigr| \leqslant K \tag{13.10}$$

and

$$\int_0^t |U_s| \mathrm{d}s \leqslant K \ \wedge\ \int_0^t Y_s^2 \mathrm{d}s \leqslant K \quad \text{a.s.} \tag{13.11}$$

This is what we will assume in the rest of the proof.

STEP 2 (Taylor's theorem): Fix $t \geqslant 0$ and a partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. Then the same derivation as that leading to (6.20) gives

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{i=1}^n \bigl[f(X_{t_i}) - f(X_{t_{i-1}})\bigr] \\ &= \sum_{i=1}^n f'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) + \frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})^2 \\ &\quad + \sum_{i=1}^n \left(\int_0^1 \Bigl[f''\bigl(\theta X_{t_{i-1}} + (1-\theta)X_{t_i}\bigr) - f''(X_{t_{i-1}}) \Bigr](1-\theta)\mathrm{d}\theta \right)(X_{t_i} - X_{t_{i-1}})^2 \end{aligned} \tag{13.12}$$

We then write (13.12) as

$$f(X_t) - f(X_0) = J_t^{(1)}(\Pi) + \cdots + J_t^{(5)}(\Pi) \tag{13.13}$$

where

$$\begin{aligned} J_t^{(1)}(\Pi) &:= \sum_{i=1}^n f'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) \\ J_t^{(2)}(\Pi) &:= \frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})(\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}}) \\ J_t^{(3)}(\Pi) &:= \frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})\Big((M_{t_i} - M_{t_{i-1}})^2 - (\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}})\Big) \\ J_t^{(4)}(\Pi) &:= \frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})\Big((X_{t_i} - X_{t_{i-1}})^2 - (M_{t_i} - M_{t_{i-1}})^2\Big) \end{aligned} \tag{13.14}$$

and

$$J_t^{(5)}(\Pi) := \sum_{i=1}^n \bigg(\int_0^1 \Big[f''\big(\theta X_{t_{i-1}} + (1-\theta)X_{t_i}\big) - f''(X_{t_{i-1}})\Big](1-\theta)\mathrm{d}\theta\bigg)(X_{t_i} - X_{t_{i-1}})^2 \tag{13.15}$$

Our aim is to show that, assuming (13.10–13.11), $J_t^{(1)}(\Pi)$ tends in probability to the first integral on the right of (13.3) while $J_t^{(2)}(\Pi)$ tends to the second integral, as $\|\Pi\| \to 0$. Then we will prove that the remaining terms vanish in probability in that limit.

STEP 3 (Limit of $J_t^{(1)}(\Pi)$): Using the analogue of (11.7) to define $\int_u^t Z_s \mathrm{d}X_s$, we start by:

Lemma 13.3 *For X a semimartingale and Z a process such that $\int_0^u Z_s \mathrm{d}X_s$ exists for all $u \leqslant t$, for all $t \geqslant u \geqslant 0$ and any random variable W,*

$$W \text{ is } \mathcal{F}_u\text{-measurable} \quad\Rightarrow\quad W\int_u^t Z_s\,\mathrm{d}X_s = \int_u^t WZ_s\,\mathrm{d}X_s \quad \text{a.s.} \tag{13.16}$$

Leaving the simple proof of this fact to homework, we now note that for X such that $\mathrm{d}X_t = U_t\mathrm{d}t + Y_t B_t$ this implies

$$\begin{aligned} \Big|J_t^{(1)}(\Pi) - \int_0^t f'(X_s)\mathrm{d}X_s\Big| &= \Big|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \big(f'(X_{t_{i-1}}) - f'(X_s)\big)\mathrm{d}X_s\Big| \\ &\leqslant \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \big|f'(X_{t_{i-1}}) - f'(X_s)\big||U_s|\mathrm{d}s + \Big|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \big(f'(X_{t_{i-1}}) - f'(X_s)\big)Y_s\mathrm{d}B_s\Big| \end{aligned} \tag{13.17}$$

For the first term on the right we get

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \big|f'(X_{t_{i-1}}) - f'(X_s)\big||U_s|\mathrm{d}s \leqslant \operatorname{osc}_{f'\circ X}\big([0,t], \|\Pi\|\big)\int_0^t |U_s|\mathrm{d}s \tag{13.18}$$

which tends to zero as $\|\Pi\| \to 0$ by the assumption (13.11) and the fact that $f' \circ X$ is continuous. Thanks to the Itô isometry enabled by (13.11), the second moment of the second term on the right of (13.17) equals the expectation of

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \big(f'(X_{t_{i-1}}) - f'(X_s)\big)^2 Y_s^2\mathrm{d}s \tag{13.19}$$

This quantity is bounded pathwise by $\operatorname{osc}_{f'\circ X}([0,t],\|\Pi\|)^2\int_0^t Y_s^2\mathrm{d}s$. Since the oscillation is at most $2K$ by (13.10) while the integral is less than K by (13.11), the Bounded Convergence Theorem ensures that the second term on the right of (13.17) tends to zero in probability as $\|\Pi\|\to 0$. This proves

$$J_t^{(1)}(\Pi) \underset{\|\Pi\|\to 0}{\overset{P}{\longrightarrow}} \int_0^t f'(X_s)\mathrm{d}X_s \tag{13.20}$$

as anticipated.

STEP 4 (Limit of $J_t^{(2)}(\Pi)$): Since $t\mapsto\langle X\rangle_t$ is non-decreasing continuous while $t\mapsto f''(X_t)$ is continuous, the standard criterion for the existence of the Stieltjes integral gives

$$J_t^{(2)}(\Pi) \underset{\|\Pi\|\to 0}{\longrightarrow} \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}\langle X\rangle_s \tag{13.21}$$

where the limit is for every path of X.

STEP 5 (Limit of $J_t^{(4)}(\Pi)$): Since $X_t=M_t+A_t$, for M and A as in (13.6), we can write

$$J_t^{(4)}(\Pi)=\frac{1}{2}\sum_{i=1}^n f''(X_{t_{i-1}})(A_{t_i}-A_{t_{i-1}})^2+\sum_{i=1}^n f''(X_{t_{i-1}})(A_{t_i}-A_{t_{i-1}})(M_{t_i}-M_{t_{i-1}}) \tag{13.22}$$

In light of (13.10), the first term on the right is bounded by (dropping $1/2$ for brevity)

$$K\,V_t^{(2)}(A,\Pi)\leqslant K\operatorname{osc}_A\big([0,t],\|\Pi\|\big)\,V_t^{(1)}(A,\Pi) \tag{13.23}$$

Since $V_t^{(1)}(A,\Pi)\leqslant\int_0^t|U_s|\mathrm{d}s\leqslant K$, this tends to zero as $\|\Pi\|\to 0$ in a pathwise sense. The second term on the right is treated similarly; first we invoke the Cauchy-Schwarz inequality along with (13.10) to bound it by the square root of

$$K^2\,V_t^{(2)}(A,\Pi)\,V_t^{(2)}(M,\Pi) \tag{13.24}$$

Taking a sequence of partitions $\{\Pi_n\}_{n\geqslant 1}$ with $\|\Pi_n\|\to 0$, we have $V_t^{(2)}(M,\Pi_n)\overset{P}{\to}\langle M\rangle_t$ by Theorem 12.5, while the remaining part of the expression tends to zero by the reasoning following (13.23). We conclude that

$$J_t^{(4)}(\Pi_n) \underset{\|\Pi_n\|\to 0}{\overset{P}{\longrightarrow}} 0 \tag{13.25}$$

as anticipated.

STEP 6 (Limit of $J_t^{(5)}(\Pi)$): Moving to the "Taylor remainder" term, here we note that, in light of (13.10),

$$\big|J_t^{(5)}(\Pi)\big|\leqslant\operatorname{osc}_{f''}\Big([-K,K],\operatorname{osc}_X\big([0,t],\|\Pi\|\big)\Big)V_t^{(2)}(X,\Pi) \tag{13.26}$$

Note that, along any sequence $\{\Pi_n\}_{n\geqslant 1}$ of partitions of $[0,t]$ with $\|\Pi_n\|\to 0$, the term $V_t^{(2)}(X,\Pi_n)$ converges in probability while the double oscillation tends to zero by continuity of f'' and X. This shows

$$J_t^{(5)}(\Pi_n) \underset{\|\Pi_n\|\to 0}{\overset{P}{\longrightarrow}} 0 \tag{13.27}$$

This term thus does not contribute in the limit.

STEP 7 (Limit of $J_t^{(3)}(\Pi)$): Addressing finally the term that is hardest to control, we note that $\langle X\rangle_t = \langle M\rangle_t$ and so

$$(M_{t_i} - M_{t_{i-1}})^2 - (\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}}) \tag{13.28}$$

is a martingale increment (see Lemma 12.7). Hence $EJ_t^{(3)}(\Pi) = 0$ and

$$\begin{aligned} E\bigl(J_t^{(3)}(\Pi)^2\bigr) &= \sum_{i=1}^n E\left(f''(X_{t_{i-1}})^2\Bigl((M_{t_i} - M_{t_{i-1}})^2 - (\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}})\Bigr)^2\right) \\ &\leqslant 2K^2 E\Bigl(V_t^{(4)}(M,\Pi) + V_t^{(2)}\bigl(\langle X\rangle,\Pi\bigr)\Bigr) \end{aligned} \tag{13.29}$$

where we used that $(a-b)^2 \leqslant 2a^2 + 2b^2$ along with the bound in (13.10) in the second step. For the second term under expectation we note that

$$V_t^{(2)}\bigl(\langle X\rangle,\Pi\bigr) \leqslant \operatorname{osc}_{\langle X\rangle}\bigl([0,t],\|\Pi\|\bigr)\, V_t^{(1)}\bigl(\langle X\rangle,\Pi\bigr) \leqslant K \operatorname{osc}_{\langle X\rangle}\bigl([0,t],\|\Pi\|\bigr) \tag{13.30}$$

where bound the first variation by the second integral in (13.11). Since $\langle X\rangle$, and thus also the oscillation in (13.30) is bounded, the expectation of this term tends to zero as $\|\Pi\| \to 0$ by the Bounded Convergence Theorem.

Concerning the fourth-variation term in (13.29), here we similarly note that

$$V_t^{(4)}(M,\Pi) \leqslant \operatorname{osc}_M\bigl([0,t],\|\Pi\|\bigr)^2\, V_t^{(2)}(M,\Pi) \tag{13.31}$$

The oscillation is bounded by $2K$ thanks to (13.10) and it tends to zero as $\|\Pi\| \to 0$ by continuity of M. To get the same convergence under expectation, we need that, under above assumptions,

$$\Bigl\{V_t^{(2)}(M,\Pi)\colon \Pi = \text{partition of } [0,t]\Bigr\} \text{ is uniformly integrable} \tag{13.32}$$

This follows from:

Lemma 13.4 *Let $\{M_t\colon t \geqslant 0\}$ be a martingale such that $|M_t| \leqslant K$ for all $t \geqslant 0$. Then for any $t \geqslant 0$ and any partition Π of $[0,t]$,*

$$E\bigl(V_t^{(2)}(M,\Pi)^2\bigr) \leqslant 48K^4 \tag{13.33}$$

Postponing the proof of the lemma until after the present proof is completed, we conclude that $EV_t^{(4)}(M,\Pi) \to 0$ as $\|\Pi\| \to 0$ and thus also $J_t^{(3)}(\Pi) \xrightarrow{P} 0$ as $\|\Pi\| \to 0$. Jointly with (13.13), (13.20), (13.21), (13.25) and (13.27), this yields the desired claim. $\square$

It remains to give:

Proof of Lemma 13.4. Writing $\Pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$ and denoting by $\{\mathcal{F}_t\}_{t\geqslant 0}$ the filtration under which M is a martingale, we first note that, for each $i = 1,\dots,n$,

$$E\Bigl(\sum_{j=i+1}^n (M_{t_j} - M_{t_{j-1}})^2 \Bigm| \mathcal{F}_{t_i}\Bigr) = E\bigl((M_t - M_{t_i})^2 \bigm| \mathcal{F}_{t_i}\bigr) \leqslant 4K^2 \tag{13.34}$$

Similarly we note that

$$E\big(V_t^{(4)}(M,\Pi)\big) \leqslant 4K^2\, E\big(V_t^{(2)}(M,\Pi)\big) = 4K^2 E\big((M_t - M_0)^2\big) \leqslant 16K^4 \tag{13.35}$$

From these we get

$$\begin{aligned} E\big(V_t^{(2)}(M,\Pi)^2\big) &= E\big(V_t^{(4)}(M,\Pi)\big) + 2\sum_{i=1}^n E\bigg((M_{t_i} - M_{t_{i-1}})^2 \sum_{j=i+1}^n (M_{t_j} - M_{t_{j-1}})^2\bigg) \\ &\leqslant 16K^2 + 8K^2 E\big(V_t^{(2)}(M,\Pi)\big) \end{aligned} \tag{13.36}$$

Bounding $EV_t^{(2)}(M,\Pi) \leqslant 4K^2$ as in (13.35), the claim follows. □

The above formula was derived for a function of X_t alone, but it extends to other functions as well. The result can be encoded by the following rules for infinitesimal differentials

$$(\mathrm{d}t)^2 = 0, \quad (\mathrm{d}t)(\mathrm{d}B_t) = 0 \quad \text{but} \quad (\mathrm{d}B_t)^2 = \mathrm{d}t \tag{13.37}$$

Using these rules, we can take the differential of the process

$$Z_t := \exp\Big\{\int_0^t Y_s\,\mathrm{d}B_s - \frac{1}{2}\int_0^t Y_s^2\,\mathrm{d}s\Big\} \tag{13.38}$$

for $Y \in \mathcal{V}^{\mathrm{loc}}$ to find that

$$\mathrm{d}Z_t = Z_t(Y_t\,\mathrm{d}B_t - \tfrac{1}{2}Y_t^2\,\mathrm{d}t) + \tfrac{1}{2}Z_tY_t^2\,\mathrm{d}t = Z_tY_t\,\mathrm{d}B_t \tag{13.39}$$

In integral form, this allows us to write Z as

$$Z_t = 1 + \int_0^t Z_sY_s\,\mathrm{d}B_s \tag{13.40}$$

showing that Z is a local martingale. Note that Z generalizes the Brownian exponential martingale $\{\mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t} \colon t \geqslant 0\}$.

As it turns out, the Itô formula extends even to functions of multiple X's that arise from a multiple of Brownian motions. The statement, which we leave without proof, is as follows:

Theorem 13.5 *Let $B^{(1)}, \dots, B^{(d)}$ be independent standard Brownian motions with a common Brownian filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Let $X = (X^{(1)}, \dots, X^{(m)})$ be and $\mathbb{R}^m$-valued stochastic process whose coordinates are given in differential form by*

$$\mathrm{d}X_t^{(i)} = U_t^{(i)}\mathrm{d}t + \sum_{k=1}^d Y_t^{(i,k)}\mathrm{d}B_t^{(k)} \tag{13.41}$$

where $\{U^{(i)} \colon i = 1,\dots,m\}$ and $\{Y^{(i,k)} \colon i = 1,\dots,m \wedge k = 1,\dots,d\}$ are as specified in Definition 12.1 for the above to make sense. Then for each $f \colon [0,\infty) \times \mathbb{R}^m \to \mathbb{R}$ that is C^1 in the *time variable and C^2 in the spatial variables and each $t \geqslant 0$,*

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial s}(s,X_s)\mathrm{d}s + \sum_{i=1}^m \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)U_s^{(i)}\mathrm{d}s$$
$$+ \sum_{i=1}^k \sum_{k=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)Y_s^{(i,k)}\mathrm{d}B_s^{(k)} + \frac{1}{2}\sum_{i,j=1}^m \sum_{k=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)Y_s^{(i,k)}Y_s^{(j,k)}\mathrm{d}s \tag{13.42}$$

As a further generalization, one can even consider the case when the Brownian motions $B^{(1)},\dots,B^{(d)}$ are not independent meaning that the covariance of this vector is not a multiple of the identity matrix. This changes (13.42) only in the last term, which in turn becomes

$$\frac{1}{2}\sum_{i,j=1}^m \sum_{k,k'=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)Y_s^{(i,k)}Y_s^{(j,k)}C_{k,k'}\mathrm{d}s \tag{13.43}$$

where $C_{k,k'} = \mathrm{Cov}(B_1^{(k)}, B_1^{(k')})$. We leave the details to the reader.

To demonstrate an example of where the multidimensional setting is particularly useful, recall that the d-dimensional standard Brownian motion is the $\mathbb{R}^d$-valued process $B = (B^{(1)},\dots,B^{(d)})$ where $B^{(1)},\dots,B^{(d)}$ are independent (1-dimensional) standard Brownian motions. The radial variable is then denoted as

$$R_t := \Big(\sum_{i=1}^d [B_t^{(i)}]^2\Big)^{1/2} \tag{13.44}$$

We can interpret this as $R_t = f \circ B$ for $f(x_1,\dots,x_d) := (x_1^2 + \dots + x_d^2)^{1/2}$. This function is not C^2 at the origin but if we stop the process before hitting the origin, Theorem 13.5 can be applied. Since, for each $k = 1,\dots,d$,

$$\frac{\partial f}{\partial x_k} = \frac{x_k}{f(x)} \quad \wedge \quad \frac{\partial^2 f}{\partial x_k^2} = \frac{1}{f(x)} - \frac{x_k^2}{f(x)^3} \tag{13.45}$$

a calculation shows that R admits the differential form

$$\mathrm{d}R_t = \frac{d-1}{2R_t}\mathrm{d}t + \frac{1}{R_t}\sum_{i=1}^d B_t^{(i)}\mathrm{d}B_t^{(i)} \tag{13.46}$$

We will see that, with proper rewrite of the second term on the right hand side, this becomes an equation for the so called *d-dimensional Bessel process*.

Further reading: Section 3.3A of Karatzas-Shreve