11. Properties of Itô integral

We now move to a discussion of the properties of the Itô integral defined above.

11.1 Linearity and additivity in domain of integration.

Our first observation is that this integral is linear in the integrand.

Lemma 11.1 (Linearity) *For all $Y, \widetilde{Y} \in \mathcal{V}^{\text{loc}}$ and all $\alpha, \beta \in \mathbb{R}$,*

$$\alpha Y + \beta \widetilde{Y} \in \mathcal{V}^{\text{loc}} \tag{11.1}$$

and, for all $t \geqslant 0$,

$$\int_0^t (\alpha Y_s + \beta \widetilde{Y}_s)\, \mathrm{d}B_s = \alpha \int_0^t Y_s \mathrm{d}B_s + \beta \int_0^t \widetilde{Y}_s \mathrm{d}B_s \quad \text{a.s.} \tag{11.2}$$

Proof. The joint measurability and adaptedness of $\alpha Y + \beta \widetilde{Y}$ is clear. For local square integrability in time, use the formula $(a+b)^2 \leqslant 2a^2 + 2b^2$ to get

$$\int_0^t (\alpha Y_s + \beta \widetilde{Y}_s)^2\, \mathrm{d}s \leqslant 2\alpha^2 \int_0^t Y_s^2 \mathrm{d}s + 2\beta^2 \int_0^t \widetilde{Y}_s^2 \mathrm{d}s \tag{11.3}$$

thus proving (11.1).

Concerning (11.2), recall that in Lemma 8.3 we observed that this holds for simple processes. The Itô isometry then extends this to all $Y, \widetilde{Y} \in \mathcal{V}$. The only subtle point is localization. Here we will rely on Theorem 10.10 and use

$$T_N := \inf\Big\{ t \geqslant 0 \colon \int_0^t (Y_s^2 + \widetilde{Y}_s^2)\mathrm{d}s \geqslant M \Big\} \tag{11.4}$$

as the family of stopping times to extend the integral to locally integrable processes. This choice ensures that $Y^{(N)} := \{Y_s 1_{\{T_N > s\}} \colon s \geqslant 0\} \in \mathcal{V}$ and $\widetilde{Y}^{(N)} := \{\widetilde{Y}_s 1_{\{T_N > s\}} \colon s \geqslant 0\} \in \mathcal{V}$ for all $N \geqslant 0$. As $T_N \to \infty$ a.s. for $Y, \widetilde{Y} \in \mathcal{V}^{\text{loc}}$, the identity (11.2) for $\alpha Y^{(N)} + \beta \widetilde{Y}^{(N)}$ yields the same identity for $\alpha Y + \beta \widetilde{Y}$. □

Another property worthy of noting is additivity under subdivision of the integration domain. The precise formulation is somewhat cumbersome due the dependence of the integral on the underlying Brownian motion. But the result is the same of what we would expect from the ordinary Stieltjes integral.

Lemma 11.2 *For $Y \in \mathcal{V}^{\text{loc}}$ associated with Brownian motion B and Brownian filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ and $u > 0$, denote*

$$\widetilde{Y}_s := Y_{u+s},\ \widetilde{B}_s := B_{u+s} - B_u \quad \text{and } \widetilde{\mathcal{F}}_s := \mathcal{F}_{u+s}. \tag{11.5}$$

Then $\{\widetilde{\mathcal{F}}_t\}_{t \geqslant 0}$ is a Brownian filtration for Brownian motion $\widetilde{B}$ and $\widetilde{Y} \in \widetilde{\mathcal{V}^{\text{loc}}}$, for $\widetilde{\mathcal{V}^{\text{loc}}}$ defined using the filtration $\{\widetilde{\mathcal{F}}_t\}_{t \geqslant 0}$. Moreover, for each $t \geqslant u$,

$$\int_0^t Y_s\, \mathrm{d}B_s = \int_0^u Y_s\, \mathrm{d}B_s + \int_0^{t-u} \widetilde{Y}_s\, \mathrm{d}\widetilde{B}_s \quad \text{a.s.} \tag{11.6}$$

We leave the proof of this lemma to homework. In order to make the notation easier, for each $Y \in \mathcal{V}^{\rm loc}$ and $0 \leqslant u \leqslant t$ we denote

$$\int_u^t Y_s \,\mathrm{d}B_s := \int_0^t Y_s \,\mathrm{d}B_s - \int_0^u Y_s \,\mathrm{d}B_s. \tag{11.7}$$

Then (11.6) shows that this is stochastic integral for the shifted Brownian path and integrand. With this notation the stochastic integral behaves very much like the ordinary Stieltjes integral. The restriction to a subinterval is naturally accompanied with:

Lemma 11.3 (Conditional Itô isometry) *Writing $\{\mathcal{F}_t\}_{t\geqslant 0}$ for the Brownian filtration underlying the definition of the stochastic integral such that $\mathcal{F}_0$ contains all P-null sets. Prove that (for a suitable version of Itô integral)*

$$\forall Y \in \mathcal{V}^{\rm loc}\,\forall t \geqslant 0\colon \quad \int_0^t Y_s \mathrm{d}B_s \;\text{ is } \mathcal{F}_t\text{-measurable} \tag{11.8}$$

and for each $Y \in \mathcal{V}$ and each $t \geqslant u \geqslant 0$,

$$E\biggl(\Bigl(\int_u^t Y_s \,\mathrm{d}B_s\Bigr)^2 \,\bigg|\, \mathcal{F}_u\biggr) = E\biggl(\int_u^t Y_s^2 \,\mathrm{d}s \,\bigg|\, \mathcal{F}_u\biggr) \quad \text{a.s.} \tag{11.9}$$

Proof (sketch). These are checked readily for Y simple. Itô isometry then extends these to to $Y \in \mathcal{V}$; localization then to $Y \in \mathcal{V}^{\rm loc}$. Details left to homework. □

11.2 Dependence on integrand.

Our next set of observations concern the dependence of the integral on the integrand Y. We start with a simple fact:

Lemma 11.4 *Writing λ for the Lebesgue measure on $[0,\infty)$, for all $Y, Y \in \mathcal{V}$ and $t \geqslant 0$,*

$$\int_0^t Y_s \,\mathrm{d}B_s = \int_0^t \widetilde{Y}_s \,\mathrm{d}B_s \quad \text{a.s.} \tag{11.10}$$

is equivalent to

$$\lambda\bigl(\{s \in [0,t]\colon\; Y_s \neq \widetilde{Y}_s\}\bigr) = 0 \quad \text{a.s.} \tag{11.11}$$

(The measure depends on ω via Y and $\widetilde{Y}$.)

Proof. By the Itô isometry, (11.10) is equivalent to $\|Y - \widetilde{Y}\|_{L^2([0,t]\times\Omega)} = 0$ which is then equivalent to (11.11). □

The Itô integral on $\mathcal{V}$ thus determines the integrand uniquely. It turns out that the same works for $Y \in \mathcal{V}^{\rm loc}$, but the proof of this requires techniques we do not have just yet. However, we can certainly prove the following intuitive variants that come (for later reference) in two lemmas:

Lemma 11.5 *Writing λ for the Lebesgue measure on $[0,\infty)$, for all $Y, Y \in \mathcal{V}^{\rm loc}$ and $t \geqslant 0$,*

$$\int_0^t Y_s \,\mathrm{d}B_s = \int_0^t \widetilde{Y}_s \,\mathrm{d}B_s \tag{11.12}$$

holds almost surely on the set

$$\Big\{\omega\in\Omega\colon \lambda\big(\{s\in[0,t]\colon\ Y_s(\omega)\neq\widetilde{Y}_s(\omega)\}\big)=0\Big\} \tag{11.13}$$

The converse direction comes in:

Lemma 11.6 *Writing λ for the Lebesgue measure on $[0,\infty)$, for all $Y, Y\in\mathcal{V}^{\rm loc}$ and $t\geqslant 0$,*

$$\lambda\big(\{s\in[0,t]\colon\ Y_s(\omega)\neq\widetilde{Y}_s(\omega)\}\big)=0 \tag{11.14}$$

holds almost surely on the set where

$$\forall 0\leqslant u\leqslant t\colon\qquad \int_0^u Y_s\,\mathrm{d}B_s=\int_0^u \widetilde{Y}_s\,\mathrm{d}B_s \tag{11.15}$$

where continuous versions of the integrals are used on the right-hand side.

We leave the proof of these lemmas to homework. The above considerations also lead to the question of whether the stochastic integral is continuous in the integrand in any reasonable sense. For $Y\in\mathcal{V}$, the continuity is in L^2-sense by Itô isometry. As it turns out, for $\mathcal{V}^{\rm loc}$ the same holds under convergence in probability.

Lemma 11.7 *Let $\{Y^{(n)}\}_{n\in\mathbb{N}}\in(\mathcal{V}^{\rm loc})^{\mathbb{N}}$ and $Y\in\mathcal{V}^{\rm loc}$ be such that, for some $t\geqslant 0$,*

$$\int_0^t [Y_s^{(n)}-Y_s]^2\mathrm{d}s\ \underset{n\to\infty}{\overset{P}{\longrightarrow}}\ 0 \tag{11.16}$$

Then

$$\int_0^t Y_s^{(n)}\,\mathrm{d}B_s\ \underset{n\to\infty}{\overset{P}{\longrightarrow}}\ \int_0^t Y_s\,\mathrm{d}B_s \tag{11.17}$$

We again leave the straightforward proof of this lemma to homework.

11.3 Riemann sums and Itô formula.

As our next item of concern, we return to the discussion we used to motivate the introduction of the stochastic integral. In particular, in Lemma 6.5 we showed that the left-endpoint Riemann sums associated with the stochastic integral of functions of the underlying Brownian motion converge in probability when the mesh of the partition tends to zero. We now extend this to all continuous processes in $\mathcal{V}^{\rm loc}$ and identify the limit object with the stochastic integral defined above.

Lemma 11.8 *Let $Y\in\mathcal{V}^{\rm loc}$ be continuous. Then for any $t\geqslant 0$ and any sequence $\{\Pi_n\}_{n\geqslant1}$ of partitions of $[0,t]$, where $\Pi_n=\{0=t_0^{(n)}<t_1^{(n)}<\dots<t_{m(n)}^{(n)}=t\}$,*

$$\|\Pi_n\|\to 0\quad\Rightarrow\quad \sum_{i=1}^{m(n)} Y_{t_{i-1}^{(n)}}\big(B_{t_i^{(n)}}-B_{t_{i-1}^{(n)}}\big)\ \underset{n\to\infty}{\overset{P}{\longrightarrow}}\ \int_0^t Y_s\,\mathrm{d}B_s \tag{11.18}$$

Proof. Define

$$Y_s^{(n)} := \sum_{i=1}^{m(n)} Y_{t_{i-1}^{(n)}} 1_{(t_{i-1}^{(n)}, t_i^{(n)}]}(s) \tag{11.19}$$

Then $Y^{(n)} \in \mathcal{V}_0$ and note that

$$\sum_{i=1}^{m(n)} Y_{t_{i-1}^{(n)}} (B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}) = \int_0^t Y_s^{(n)} \, \mathrm{d}B_s \tag{11.20}$$

A straightforward computation shows

$$\int_0^t [Y_s^{(n)} - Y_s]^2 \mathrm{d}s \leqslant \mathrm{osc}_Y\big([0,t], \|\Pi_n\|\big)^2 t \tag{11.21}$$

and the oscillation tends to zero once $\|\Pi_n\| \to 0$ by the assumed continuity of Y. This implies that (11.16) is in force; the conclusion (11.17) then yields the claim. □

Having settled the convergence of Riemann sums, we can now use the derivation underlying Lemma 6.5 to get:

Theorem 11.9 (Itô formula) *Let $f \in C^2(\mathbb{R})$. Then for each $t \geqslant 0$,*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) \mathrm{d}B_s + \frac{1}{2} \int_0^t f''(B_s) \mathrm{d}s \tag{11.22}$$

Proof. First observe that both integrals in (11.22) exist by the fact that $t \mapsto f'(B_t)^2$ and $t \mapsto |f''(B_t)|$ are both continuous and locally integrable. Given a partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, the proof of Lemma 6.5 showed that

$$f(B_t) - f(B_0) = \sum_{i=1}^n f'(B_{t_{i-1}})(B_{t_i)} - B_{t_{i-1}}) + \frac{1}{2} \sum_{i=1}^n f''(B_{t_{i-1}})(t_i - t_{i-1}) + \epsilon(\Pi) \tag{11.23}$$

where $\epsilon(\Pi)$ is a random quantity that obeys $\epsilon(\Pi) \xrightarrow{P} 0$ whenever $\|\Pi\| \to 0$. Since $t \mapsto f''(B_t)$ is continuous, the second sum converges to the second integral in (11.22) by the usual criteria in Riemann integration theory. Lemma 11.8 gives convergence of the first sum to the stochastic integral in (11.22). □

We remark that the condition that f is twice continuously differentiable can be weakened to f' being absolutely continuous, and thus differentiable in the Lebesgue sense (so that the Fundamental Theorem of Calculus can be invoked).

Recall that the quantity involving the integrals in (11.22) is called the Stratonovich integral $\int_0^t f'(B_s) \circ \mathrm{d}B_s$; see (6.41). The advantage of the Stratonovich integral is that (at least formally) the Fundamental Theorem of Calculus holds. However, a major drawback is that one needs to be able to differentiate f in order to integrate $f \circ B$. Another disadvantage is that there is no obvious extension from integrals of $\{f(B_s) \colon s \geqslant 0\}$ to those of more general processes $\{Y_s \colon s \geqslant 0\}$. Notwithstanding, the Stratonovich integral is useful because is appears in some natural practical considerations:

Theorem 11.10 (Wong and Zakai 1969) *Let $\{B_t^{(n)} : t \geqslant 0\}_{n\in\mathbb{N}}$ be stochastic processes and let $\{B_t : t \geqslant 0\}$ be a Brownian motion such that, for some $t \geqslant 0$,*

(1) $\forall n \in \mathbb{N} \colon B^{(n)}$ *is continuous on* $[0,t] \,\wedge\, V_t^{(1)}(B^{(n)}) < \infty$ *a.s.*
(2) $B_0^{(n)} \xrightarrow[n\to\infty]{P} B_0$ *and* $B_t^{(n)} \xrightarrow[n\to\infty]{P} B_t$.

Then for all $f \in C^1(\mathbb{R})$,

$$\int_0^t f(B_s^{(n)}) \mathrm{d}B_s^{(n)} \xrightarrow[n\to\infty]{P} \int_0^t f(B_s) \circ \mathrm{d}B_s \tag{11.24}$$

Here, on the left, the integral is in the Riemann-Stieltjes sense.

Proof. The assumed continuity and bounded variation of $B^{(n)}$ ensures that the Stieltjes integral exists. Moreover, writing F for an antiderivative of f, we have

$$\int_0^t f(B_s^{(n)}) \mathrm{d}B_s^{(n)} = F(B_t^{(n)}) - F(B_0^{(n)}) \tag{11.25}$$

In light of (2), the right hand side converges in probability to

$$F(B_t) - F(B_0) = \int_0^t f(B_s) \circ \mathrm{d}B_s. \tag{11.26}$$

This gives the claim. □

While the previous result is not very deep mathematically, it is a warning sign that simulating the Itô integral directly on a computer may require considerable care.

Further reading: Section 3.2 in Øksendal