10. Extension via localization

In the previous lectures we gradually defined the Itô integral for every process that is jointly-measurable, adapted and square integrable with respect to both time and the underlying probability measure on any compact interval of times. While technically convenient for the construction, the requirement of square integrability is often a nuisance and so we will now show how to get even beyond that.

Our standing assumption is that we are given a probability space $(\Omega, \mathcal{F}, P)$ that supports a standard Brownian motion $\{B_t \colon t \geqslant 0\}$ and a Brownian filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ such that $\mathcal{F}_0$ contains all P-null sets. Let $\mathcal{V}^{\text{loc}}$ be the class of processes $Y = \{Y_t \colon t \geqslant 0\}$ on $(\Omega, \mathcal{F}, P)$ that are jointly measurable, adapted to $\{\mathcal{F}_t\}_{t \geqslant 0}$ and obey

$$\forall t \geqslant 0\colon \qquad \int_0^t Y_s^2 \, \mathrm{d}s < \infty \quad \text{a.s.} \tag{10.1}$$

where the integral is meaningful (as a Lebesgue integral) thanks to the assumption of joint measurability (which implies that $t \mapsto Y_t(\omega)$ is a Borel function for each $\omega \in \Omega$). The said extension now comes in:

Theorem 10.1 *Let $Y \in \mathcal{V}^{\text{loc}}$ and, for $M > 0$, set*

$$T^{(M)} := \inf\Big\{ t \geqslant 0 \colon \int_0^t Y_s^2 \, \mathrm{d}s \geqslant M \Big\} \tag{10.2}$$

Then for each $M > 0$,

(1) $\{Y_t 1_{\{T^{(M)} > t\}} \colon t \geqslant 0\} \in \mathcal{V}$, *and*
(2) *for all $t \geqslant 0$ and all $N \geqslant M$,*

$$\int_0^t Y_s 1_{\{T^{(N)} > s\}} \, \mathrm{d}B_s = \int_0^t Y_s 1_{\{T^{(M)} > s\}} \, \mathrm{d}B_s \quad \text{a.s. on } \{T^{(M)} > t\} \tag{10.3}$$

In particular, for each $t \geqslant 0$,

$$\int_0^t Y_s \, \mathrm{d}B_s := \lim_{M \to \infty} \int_0^t Y_s 1_{\{T^{(M)} > s\}} \, \mathrm{d}B_s \tag{10.4}$$

exists and is finite a.s.

The main difficulty in part (1) is to show that the process there is adapted and measurable. For this we recall the following concept:

Definition 10.2 *Given a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$, a $[0, \infty]$-valued random variable T is a* stopping time *if*

$$\forall t \geqslant 0\colon \qquad \{T \leqslant t\} \in \mathcal{F}_t \tag{10.5}$$

We now note:

Lemma 10.3 *Under the above setting and assumptions, $T^{(M)}$ in* (10.2) *is a stopping time for each $M \geqslant 0$ and each $Y \in \mathcal{V}^{\text{loc}}$.*

Proof. A standard monotone class argument shows that $\int_0^u Y_s^2\,\mathrm{d}s$ is $\mathcal{F}_u$-measurable once Y is jointly measurable and adapted. To get the claim form this we note that, by the continuity of $u \mapsto \int_0^u Y_s^2\,\mathrm{d}s$ on the set where it is finite, we have

$$\{T^{(M)} \leqslant t\} = \left\{\int_0^t Y_s^2\,\mathrm{d}s \geqslant M\right\} \cup \left(\left\{\int_0^t Y_s^2\,\mathrm{d}s < M\right\} \cap \bigcap_{n\geqslant 1}\left\{\int_0^{t+1/n} Y_s^2\,\mathrm{d}s = \infty\right\}\right) \tag{10.6}$$

The events in the giant intersections are P-null and so belong to $\mathcal{F}_0$ by assumption. The remaining events lie in $\mathcal{F}_t$ and so we get $\{T^{(M)} \leqslant t\} \in \mathcal{F}_t$ as desired. □

The main argument of the proof of Theorem 10.2 now comes in:

Proposition 10.4 *Under the aforementioned standing assumptions, for each $Y \in \mathcal{V}$ and each stopping time T we have*

$$\left\{Y_t 1_{\{T>t\}} \colon t \geqslant 0\right\} \in \mathcal{V} \tag{10.7}$$

and, writing $\{I_t \colon t \geqslant 0\}$ for a continuous version of $t \mapsto \int_0^t Y_s \mathrm{d}B_s$, for each $t \geqslant 0$,

$$\int_0^{T\wedge t} Y_s\,\mathrm{d}B_s := I_{T\wedge t} = \int_0^t Y_s 1_{\{T>s\}}\,\mathrm{d}B_s \quad \text{a.s.} \tag{10.8}$$

Here the middle quantity is I_u evaluated at $u := T \wedge t$.

The proof is a blueprint for many similar proofs involving stopping times. First we check that the result holds for discrete-valued stopping times and simple processes. Then we take limits to extend this to the stated general case.

We start with the discretization of T. For integer $N \geqslant 0$, define

$$T_N := 2^{-N}\lceil 2^N T\rceil. \tag{10.9}$$

We then have:

Lemma 10.5 *T_N is a stopping time with $T_N \downarrow T$ as $N \to \infty$. The processes $\{1_{\{T_N>t\}} \colon t \geqslant 0\}$ with $N \geqslant 0$, as well as $\{1_{\{T>t\}} \colon t \geqslant 0\}$, are adapted and jointly measurable.*

Proof. Since $T_N = k2^{-N}$ on $\{(k-1)2^{-N} < T \leqslant k2^{-N}\}$, we have

$$\{T_N \leqslant t\} = \bigcup_{k\leqslant 2^N t}\left\{(k-1)2^{-N} < T \leqslant k2^{-N}\right\} = \left\{T \leqslant 2^{-N}\lfloor 2^N t\rfloor\right\} \tag{10.10}$$

It follows that if T is a stopping time, then so is T_N. The dyadic nature of the discretization implies that $\{T_N\}_{N\geqslant 0}$ is non-increasing and, in light of

$$T_N - 2^{-N} \leqslant T \leqslant T_N \tag{10.11}$$

we get $T_N \downarrow T$ as $N \to \infty$ (including on $\{T = \infty\} = \{T_N = \infty\}$).

By the fact that $\{T_N\}_{N\geqslant 0}$ and T are stopping times, the processes $\{1_{\{T_N>t\}} \colon t \geqslant 0\}$ for any $N \geqslant 0$, as well as $\{1_{\{T>t\}} \colon t \geqslant 0\}$ are adapted. As to joint measurability, note that given any random variable U taking values in a discrete set $\{u_i\colon i\geqslant 0\}$, the representation $1_{\{U>t\}}=\sum_{i\geqslant 0}1_{(u_i,\infty)}(t)1_{\{U=u_i\}}$ shows that $\{1_{\{U>t\}}\colon t\geqslant 0\}$ is jointly measurable. Setting $U_n:=2^{-n}\lfloor 2^nU\rfloor$ yields a sequence of discrete-valued random variables with $U_n\uparrow U$. As $1_{U_n>t}\uparrow 1_{U>t}$ for each $t\geqslant 0$, the process $\{1_{\{U>t\}}\colon t\geqslant 0\}$ is jointly measurable for all random variables U. $\square$

Next, since $Y\in\mathcal{V}$, Theorem 9.5 tells us that there are $\{Y^{(n)}\}_{n\geqslant 0}\in\mathcal{V}_0^{\mathbb{N}}$ such that

$$[\![\, Y-Y^{(n)}]\!]\underset{n\to\infty}{\longrightarrow}0. \tag{10.12}$$

We now proceed to restate and prove the claim for the discretized processes:

Lemma 10.6 *For each $n\geqslant 1$ and $N\geqslant 0$,*

$$\bigl\{Y_t^{(n)}1_{\{T_N>t\}}\colon t\geqslant 0\bigr\}\in\mathcal{V} \tag{10.13}$$

and, for each $t\geqslant 0$,

$$\int_0^{T_N\wedge t}Y_s^{(n)}\,\mathrm{d}B_s=\int_0^t Y_s^{(n)}1_{\{T_N>s\}}\,\mathrm{d}B_s\quad\text{a.s.} \tag{10.14}$$

Here, on the left, we use the defining expression of the integral for processes in V_0.

Proof. The containment $Y^{(n)}\in\mathcal{V}_0$ guarantees that each $Y^{(n)}$ is adapted, jointly measurable and bounded. Lemma 10.5 implies the same about $\{1_{\{T_N>t\}}\colon t\geqslant 0\}$ and thus also about $\{Y_t^{(n)}1_{\{T_N>t\}}\colon t\geqslant 0\}$. This is enough to ensure containment in $\mathcal{V}$.

For the second part of the claim, note that for $t\geqslant 0$ the Tonelli Theorem applied to infinite sums along with the fact that T_N takes values in $2^{-N}\mathbb{N}$ yields

$$\begin{aligned}1_{\{T_N>t\}}&=\sum_{\ell\geqslant 0}1_{\{T_N=\ell 2^{-N}\}}1_{[0,\,\ell 2^{-N})}(t)=\sum_{\ell\geqslant 0}\sum_{k=0}^{\ell-1}1_{\{T_N=\ell 2^{-N}\}}1_{[k2^{-N},(k+1)2^{-N})}(t)\\&=\sum_{k\geqslant 0}\sum_{\ell=k+1}^{\infty}1_{\{T_N=\ell 2^{-N}\}}1_{[k2^{-N},(k+1)2^{-N})}(t)=\sum_{k\geqslant 0}1_{\{T_N>k2^{-N}\}}1_{[k2^{-N},(k+1)2^{-N})}(t).\end{aligned} \tag{10.15}$$

The expression on the right has almost the form of a simple process except for two points: The intervals are open/closed at opposite ends than they should be and the sum is infinite rather than finite. To address these issues, assume that Y takes the form

$$Y_t^{(n)}=Z_0 1_{\{0\}}(t)+\sum_{i=1}^m Z_i 1_{(t_{i-1},t_i]}(t), \tag{10.16}$$

where $t_0=0<t_1<\dots<t_m$ and $Z_i\in L^\infty$ is $\mathcal{F}_{t_{i-1}}$-measurable for each i so that $t_m 2^N\in\mathbb{N}$ and $\{t_i\colon i=0,\dots,t_m\}$ contains all the points in $\{k2^{-N}\colon k\geqslant 0\wedge k2^{-N}\leqslant t_m\}$. Then

$$R_t:=\sum_{i=1}^m Z_i 1_{\{T_N>t_{i-1}\}}1_{(t_{i-1},t_i]}(t) \tag{10.17}$$

is a simple process such that

$$\left|Y_t^{(n)}1_{\{T_N>t\}} - R_t\right| \leqslant \sum_{i=0}^{m} \|Z_i\| 1_{\{t_{i-1}\}}(t) \tag{10.18}$$

where $t_{-1} := 0$. The Lebesgue integral of the right-hand side vanishes which means that the processes $\{Y_t^{(n)}1_{\{T_N>t\}} \colon t \geqslant 0\}$ and $\{R_t \colon t \geqslant 0\}$ are equivalent in $\mathcal{V}$.

Using the defining expression for integrals of simple functions, we then get

$$\int_0^{T_N\wedge t} Y_s^{(n)}\,\mathrm{d}B_s = \sum_{i=1}^{m} Z_i\big(B_{T_N\wedge t\wedge t_i} - B_{T_N\wedge t\wedge t_{i-1}}\big) \tag{10.19}$$

The above-mentioned equivalence in turn gives

$$\int_0^{t} Y_s^{(n)}1_{\{T_N>s\}}\,\mathrm{d}B_s \overset{\text{a.s.}}{=} \int_0^t R_s\,\mathrm{d}B_s = \sum_{i=1}^{m} Z_i 1_{\{T>t_{i-1}\}}\big(B_{t\wedge t_i} - B_{t\wedge t_{i-1}}\big) \tag{10.20}$$

Since T_N takes values in $\{t_i \colon i=0,\dots,m\}$, we have

$$\begin{aligned} B_{T_N\wedge t\wedge t_i} - B_{T_N\wedge t\wedge t_{i-1}} &= \big(B_{T_N\wedge t\wedge t_i} - B_{T_N\wedge t\wedge t_{i-1}}\big)1_{\{T_N\geqslant t_i\}} \\ &= \big(B_{t\wedge t_i} - B_{t\wedge t_{i-1}}\big)1_{\{T_N\geqslant t_i\}} = \big(B_{t\wedge t_i} - B_{t\wedge t_{i-1}}\big)1_{\{T_N>t_{i-1}\}} \end{aligned} \tag{10.21}$$

where the first equality follows from the fact that the difference on the left-hand side vanishes on $\{T_N < t_i\} = \{T_N \leqslant t_{i-1}\}$ and the rest is a rewrite based on the complementary equality $\{T_N \geqslant t_i\} = \{T_N > t_{i-1}\}$. This equates the right-hand sides of (10.20) with (10.19) proving (10.14) as desired. □

We now start taking limits, starting with the integral of the truncated process:

Lemma 10.7 *For each $t \geqslant 0$,*

$$\lim_{N\to\infty}\limsup_{n\to\infty} E\left(\left|\int_0^t Y_s^{(n)}1_{\{T_N>s\}}\,\mathrm{d}B_s - \int_0^t Y_s 1_{\{T>s\}}\,\mathrm{d}B_s\right|^2\right) = 0. \tag{10.22}$$

Proof. Note that (10.11) implies

$$0 \leqslant 1_{\{T_N>s\}} - 1_{\{T>s\}} = 1_{\{T\leqslant s<T_N\}} \tag{10.23}$$

The triangle inequality combined with Itô isometry gives

$$\begin{aligned} &\|Y^{(N)}1_{\{T_N>\cdot\}} - Y1_{\{T>\cdot\}}\|_{L^2([0,t]\times\Omega)} \\ &\qquad \leqslant \|Y^{(n)} - Y\|_{L^2([0,t]\times\Omega)} + \left[\int_0^t Y_s^2 1_{\{T\leqslant s<T_N\}}\mathrm{d}s\right]^{1/2} \end{aligned} \tag{10.24}$$

The first term vanishes as $n \to \infty$ by (10.12). Using $1_{\{T\leqslant s<T_N\}} \to 1_{\{T=s\}}$ as $N \to \infty$, the Dominated Convergence Theorem shows that the second term vanishes as $N \to \infty$. In light of the isometry-based construction of the Itô integral, this implies (10.22). □

Next we address the limit of the integral truncated by the stopping time:

Lemma 10.8 *For each $t \geqslant 0$ and each $N \geqslant 0$,*

$$\int_0^{T_N \wedge t} Y_s^{(n)} \, \mathrm{d}B_s \xrightarrow[n\to\infty]{P} \int_0^{T_N \wedge t} Y_s \, \mathrm{d}B_s \tag{10.25}$$

Proof. Pick $\epsilon > 0$ and note that, by Doob's L^2-inequality (see Lemma 9.3) relying on the fact that the stochastic integral is an L^2-martingale,

$$\begin{aligned} P\left(\left|\int_0^{T_N\wedge t} Y_s^{(n)} \, \mathrm{d}B_s - \int_0^{T_N\wedge t} Y_s \, \mathrm{d}B_s\right| > \epsilon\right) \\ \leqslant P\left(\sup_{0\leqslant u\leqslant t}\left|\int_0^u Y_s^{(n)} \, \mathrm{d}B_s - \int_0^u Y_s \, \mathrm{d}B_s\right| > \epsilon\right) \leqslant \frac{1}{\epsilon^2}\|Y^{(n)} - Y\|^2_{L^2([0,t]\times\Omega)} \end{aligned} \tag{10.26}$$

The right-hand side tends to zero by (10.12). □

We are now ready to give:

Proof of Proposition 10.4. The claim in (10.7) follows by the same arguments as those used in the proof of Lemma 10.6. To get (10.8), we now invoke (10.14) along with Lemmas 10.7–10.8 and the fact that

$$\int_0^{T_N\wedge t} Y_s \, \mathrm{d}B_s \xrightarrow[N\to\infty]{} \int_0^{T\wedge t} Y_s \, \mathrm{d}B_s \quad \text{a.s.} \tag{10.27}$$

implied by $T_N \downarrow T$ and the fact that the stochastic integral admits a continuous version almost surely (see Theorem 9.2). □

Moving over to the proof of the main result of this section, we first observe:

Lemma 10.9 *For each $Y \in \mathcal{V}^{\text{loc}}$ and each $M \geqslant 0$,*

$$\left\{Y_t 1_{\{T^{(M)}>t\}} \colon t \geqslant 0\right\} \in \mathcal{V} \tag{10.28}$$

Proof. Lemma 10.5 gives that the process is adapted and measurable. To check containment in $\mathcal{V}$, note $\int_0^u Y_s^2 \mathrm{d}s \leqslant M$ on $\{T^{(M)} > u\}$. Hence,

$$\int_0^t (Y_s 1_{\{T^{(M)}>s\}})^2 \, \mathrm{d}s = \lim_{u\uparrow T^{(M)}\wedge t} \int_0^u Y_s^2 \mathrm{d}s \leqslant M \tag{10.29}$$

by the Monotone Convergence Theorem. As this holds for all $t \geqslant 0$, we get (10.28). □

Proof of Theorem 10.1. Let $Y \in \mathcal{V}^{\text{loc}}$. Part (1) was proved in Lemma 10.9. For (2) we note that, since $M \mapsto T^{(M)}$ is non-decreasing,

$$N \geqslant M \quad \Rightarrow \quad 1_{\{T^{(M)}>s\}} = 1_{\{T^{(M)}>s\}} 1_{\{T^{(N)}>s\}} \tag{10.30}$$

Proposition 10.4 then gives

$$\begin{aligned} \int_0^t Y_s 1_{\{T^{(M)}>s\}} \, \mathrm{d}B_s &\overset{N\geqq M}{=} \int_0^t Y_s 1_{\{T^{(N)}>s\}} 1_{\{T^{(M)}>s\}} \, \mathrm{d}B_s \overset{\text{a.s.}}{=} \int_0^{T^{(M)}\wedge t} Y_s 1_{\{T^{(N)}>s\}} \, \mathrm{d}B_s \\ &= \int_0^t Y_s 1_{\{T^{(N)}>s\}} \, \mathrm{d}B_s \quad \text{on } \{T^{(M)} > t\} \end{aligned} \tag{10.31}$$

This proves (10.3) as well as (10.4), for (a.s.) constant sequences have (a.s.) limits and that limit exists on $\bigcup_{M\geqslant 1}\{T^{(M)} > t\}$ which is a full-measure event by (10.2). $\square$

As a consequence of Theorems 9.2, 9.5 and 10.1 and also Proposition 10.4 we get:

Theorem 10.10 *Suppose the Brownian filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is such that $\mathcal{F}_0$ contains all P-null sets. Then for all $Y \in \mathcal{V}^{\text{loc}}$ the integral defined in* (10.4) *admits a continuous version that is adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$ and obeys*

$$\int_0^{T\wedge t} Y_s \mathrm{d}B_s = \int_0^t Y_s 1_{\{T>s\}} \mathrm{d}B_s \quad \text{a.s.} \tag{10.32}$$

for each $t \geqslant 0$ and each stopping time T with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$. In particular, if $\{T_N\}_{N\geqslant 0}$ is a sequence of stopping times such that $\{Y_t 1_{\{T_N>t\}} \colon t \geqslant 0\} \in \mathcal{V}$ for each $N \geqslant 0$ and $T_N \to \infty$ a.s., then for each $t \geqslant 0$,

$$\int_0^t Y_s 1_{\{T_N>s\}} \mathrm{d}B_s \underset{N\to\infty}{\longrightarrow} \int_0^t Y_s \mathrm{d}B_s \quad \text{a.s.} \tag{10.33}$$

Proof. That a continuous version of the extended Itô integral exists follows from the fact that on $\{T^{(M)} > t\}$, the map $u \mapsto \int_0^u Y_s \mathrm{d}B_s$ for $u \in [0,t]$ coincides with $u \mapsto \int_0^u Y_s 1_{\{T^{(M)}>s\}} \mathrm{d}B_s$ which admits a continuous version by (10.28) and Theorems 9.2. Proposition 10.4 then gives

$$\int_0^{T\wedge t} Y_s \mathrm{d}B_s = \int_0^t Y_s 1_{\{T^{(M)}>s\}} 1_{\{T>s\}} \mathrm{d}B_s \quad \text{a.s. on } \{T^{(M)} > t\} \tag{10.34}$$

Letting $\widetilde{T}^{(M)} := \inf\{u \geqslant 0 \colon \int_0^u Y_s^2 1_{\{T>s\}} \mathrm{d}s \geqslant M\}$, we now check $\widetilde{T}^{(M)} \wedge T = T^{(M)} \wedge T$ and so

$$\int_0^t Y_s 1_{\{T^{(M)}>s\}} 1_{\{T>s\}} \mathrm{d}B_s = \int_0^t Y_s 1_{\{T>s\}} 1_{\{\widetilde{T}^{(M)}>s\}} \mathrm{d}B_s \tag{10.35}$$

The right-hand side equals the integral on the right of (10.32) on $\{T^{(M)} > t\}$, which is a subset of $\{\widetilde{T}^{(M)} > t\}$. Hence we get (10.32) on $\bigcup_{M\geqslant 0}\{T^{(M)} > t\}$ which is a full-measure set by the fact that $Y \in \mathcal{V}^{\text{loc}}$. By (10.32), the object on the left-hand side of (10.33) equals the right-hand side once $T_N > t$. As $T_N \to \infty$ a.s., the equality holds a.s. $\square$

Remark 10.11 The upshot of (10.33) is that the extension of the Itô integral to $Y \in \mathcal{V}^{\text{loc}}$ in (10.4) can be done along *any* sequence $\{T_N\}_{N\geqslant 0}$ of stopping times such that $T_N \to \infty$ a.s. as $N \to \infty$ and $\{Y_t 1_{\{T_N>t\}} \colon t \geqslant 0\} \in \mathcal{V}$ for every $N \geqslant 0$.

The Itô integral has now been extended to all $Y \in \mathcal{V}^{\text{loc}}$. A further extension can be attempted by assuming (10.2) (or having Y defined) only up to a stopping time. These extensions are handled by working with the *stopped process* $\{Y_{t\wedge T} \colon t \geqslant 0\}$ (which still needs to be assumed in $\mathcal{V}^{\text{loc}}$). We will also see later that no extension beyond processes for which $t \mapsto Y_t$ is locally square integrable exists.

Further reading: Section 3.2D of Karatzas-Shreve