HW#6: due Mon 3/18/2024

The following problems use the notation from class/class notes.

Problem 1: Prove the Feynman-Kac formula: Let $u \in C^{1,2}((0,\infty)\times\mathbb{R}^d)\cap C([0,\infty)\times\mathbb{R}^d)$ be bounded and such that, for some bounded continuous $V\colon \mathbb{R}^d \to \mathbb{R}$,

$$\forall t>0\,\forall x\in\mathbb{R}^d\colon\quad \frac{\partial u}{\partial t}(t,x) = \tfrac{1}{2}\Delta u(t,x) + V(x)u(t,x)$$

Then

$$u(t,x) = E^x\biggl(u(0,B_t)\exp\Bigl\{\int_0^t V(B_s)\mathrm{d}s\Bigr\}\biggr)$$

where B is a standard Brownian motion that, under E^x, is started from $B_0 = x$.

Problem 2: Let $D \subseteq \mathbb{R}^2$ be non-empty and open and assume $x_0 \in \partial D$ is such that there is a continuous injective map (a simple curve) $\gamma\colon [0,\infty) \to \mathbb{R}^2$ with

$$\gamma([0,\infty)) \subseteq \mathbb{R}^2 \smallsetminus D \;\wedge\; \gamma(0) = x_0 \;\wedge\; \lim_{t\to\infty}\gamma(t) = \infty$$

Prove that x_0 is regular.

Hint : Use that, for each annulus $A_r := \{x \in \mathbb{R}^2\colon r < |x - x_0| < 3r\}$, with positive probability that does not depend on r, the Brownian motion started at x_0 "draws" a "closed loop" inside $A_{r,R}$ after hitting the mid-circle $\{x \in \mathbb{R}^2\colon |x - x_0| = 2r\}$. Then employ stopping time arguments to show that the curve $\gamma([0,\infty))$ is hit instantaneously.

Problem 3: KARATZAS-SHREVE EX 2.26, PAGE 299

Problem 4: KARATZAS-SHREVE EX 2.27, PAGE 299

Problem 5: Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a Borel function. Prove that solutions to the SDE

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t,$$

cannot explode in finite time. More precisely, if $\{X_t\colon t \in [0,\tau_\infty)\}$ is a weak solution to this SDE and

$$\tau_n := \inf\bigl\{t \geqslant 0\colon |X_t| \geqslant n\bigr\}$$

then $\tau_\infty := \lim_{n\to\infty}\tau_n$ (called the "explosion time") obeys $\tau_\infty = \infty$ a.s.

Problem 6: Let $\theta > 1$ and consider the SDE

$$\mathrm{d}X_t = (X_t^+)^\theta\mathrm{d}t + \mathrm{d}B_t$$

Prove that any solution started from $X_0 > 0$ explodes with positive probability (in fact, with probability one).

Hint: Write the SDE in ODE form and compare it with an ODE that can be solved explicitly, assuming that B is suitably controlled via a stopping time.