HW#4: due Fri 2/16/2024

The first couple of problems deal with Itô integral on the space $\mathcal{V}^{\text{loc}}$ of jointly-measurable, adapted processes Y that are locally square integrable a.s. The rest of the exercises give some practice of the notion of (semi)martingales.

Problem 1: ØKSENDAL EX 3.2, PAGE 38

Problem 2: For $Y \in \mathcal{V}^{\text{loc}}$ associated with Brownian motion B and Brownian filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $u > 0$, denote

$$\widetilde{Y}_s := Y_{u+s},\ \widetilde{B}_s := B_{u+s} - B_u \quad \text{ and } \widetilde{\mathcal{F}}_s := \mathcal{F}_{u+s}.$$

Prove that $\{\widetilde{\mathcal{F}}_t\}_{t\geq 0}$ is a Brownian filtration for Brownian motion $\widetilde{B}$ and $\widetilde{Y} \in \widetilde{\mathcal{V}^{\text{loc}}}$, for $\widetilde{\mathcal{V}^{\text{loc}}}$ defined using the filtration $\{\widetilde{\mathcal{F}}_t\}_{t\geq 0}$. Moreover, for each $t \geq u$,

$$\int_0^t Y_s \,\mathrm{d}B_s = \int_0^u Y_s \,\mathrm{d}B_s + \int_0^{t-u} \widetilde{Y}_s \,\mathrm{d}\widetilde{B}_s \quad \text{a.s.}$$

Problem 3: Let $Y \in \mathcal{V}^{\text{loc}}$ be associated with Brownian motion B and Brownian filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which we assume is such that $\mathcal{F}_0$ contains all P-null sets. Prove that

$$\forall t \geq 0\colon \quad \int_0^t Y_s \mathrm{d}B_s \text{ is } \mathcal{F}_t\text{-measurable}$$

and, assuming $Y \in \mathcal{V}$, the conditional Itô isometry

$$E\biggl(\Bigl(\int_u^t Y_s \,\mathrm{d}B_s\Bigr)^2 \,\bigg|\, \mathcal{F}_u\biggr) = E\biggl(\int_u^t Y_s^2 \,\mathrm{d}s \,\bigg|\, \mathcal{F}_u\biggr) \quad \text{a.s.}$$

holds for all $u \in [0,1]$. Here $\int_u^t Y_s \,\mathrm{d}B_s := \int_0^t Y_s \,\mathrm{d}B_s - \int_0^u Y_s \,\mathrm{d}B_s$.

Problem 4: Prove that, for any $\{Y^{(n)}\}_{n\in\mathbb{N}} \in (\mathcal{V}^{\text{loc}})^{\mathbb{N}}$ and $Y \in \mathcal{V}^{\text{loc}}$ and any $t \geq 0$,

$$\int_0^t [Y_s^{(n)} - Y_s]^2 \mathrm{d}s \xrightarrow[n\to\infty]{P} 0$$

implies

$$\int_0^t Y_s^{(n)} \,\mathrm{d}B_s \xrightarrow[n\to\infty]{P} \int_0^t Y_s \,\mathrm{d}B_s$$

This is a statement of continuity for Itô integral on $\mathcal{V}^{\text{loc}}$.

Problem 5: Writing λ for the Lebesgue measure on $[0,\infty)$, prove that for all $Y, Y \in \mathcal{V}^{\text{loc}}$ and $t \geq 0$ (and assuming continuous versions of the integrals), the events

$$\forall u \leq t\colon \quad \int_0^u Y_s \,\mathrm{d}B_s = \int_0^u \widetilde{Y}_s \,\mathrm{d}B_s$$

and

$$\Bigl\{\omega \in \Omega\colon \lambda\bigl(\{s \in [0,t]\colon\ Y_s(\omega) \neq \widetilde{Y}_s(\omega)\}\bigr) = 0\Bigr\}$$

differ by a P-null set. (This is a variation on Øksendal ex. 3.15 page 41)

Problem 6: Let $\{M_t\colon t \geq 0\}$ be a continuous local martingale such that, for some $t \geq 0$,

$$V_t^{(1)}(M) < \infty \quad \text{a.s.}$$

Prove that then

$$P\big(\forall s \in [0,t]\colon M_s = M_0\big) = 1$$

You may use that a continuous martingale stopped at a stopping time is a martingale.

Problem 7: ØKSENDAL EX 3.12, PAGE 40

Problem 8: Let X be a semimartingale and Z an adapted, jointly measurable process such that $\int_0^t Z_s \mathrm{d}X_s$ exists for all $t \geq 0$. Prove that for all $t \geq u \geq 0$ and any random variable W,

$$W \text{ is } \mathcal{F}_u\text{-measurable} \quad \Rightarrow \quad W \int_u^t Z_s \,\mathrm{d}X_s = \int_u^t W Z_s \,\mathrm{d}X_s \quad \text{a.s.}$$

Here $\int_u^t Z_s \mathrm{d}X_s := \int_0^t Z_s \mathrm{d}X_s - \int_0^u Z_s \mathrm{d}X_s$.
