HW#2: due Fri 1/26/2024

This problem set focuses on standard Borel spaces and then measurability issues associated with families of random variables.

Problem 1: Let $(\mathscr{X}, \Sigma)$ be a standard Borel space — namely, $\mathscr{X}$ is a complete, separable metric space and Σ is the σ-algebra of its Borel sets. Prove that $(\mathscr{X}^n, \Sigma^{\otimes n})$ is standard Borel for each natural $n \geq 1$.

Problem 2: Let $C([0,\infty))$ be the space of continuous functions $[0,\infty) \to \mathbb{R}$. Define

$$\varrho(f,g) := \sum_{n\geq 1} 2^{-n} \sup_{0\leq t\leq n} |f(t) - g(t)| \wedge 1$$

Prove that ϱ is a metric that makes $C([0,\infty))$ complete and separable, and thus Polish.

Problem 3: Next, as a review, prove the following standard claims about the preimage map: Let $X\colon \Omega \to \mathscr{X}$ be a function. Then

(1) for each σ-algebra Σ on $\mathscr{X}$,

$$\big\{\{X \in B\}\colon B \in \Sigma\big\} \text{ is a } \sigma\text{-algebra on } \Omega$$

(2) for each σ-algebra $\mathcal{F}$ on Ω,

$$\big\{B\colon B \subseteq \mathscr{X} \ \wedge\ \{X \in B\} \in \mathcal{F}\big\} \text{ is a } \sigma\text{-algebra on } \mathscr{X}$$

Problem 4: Let T be a set and let $\{X_t\colon t \in T\}$ be a family of random variables. Prove

$$\sigma(X_t\colon t \in T) = \bigcup_{\substack{S\subset T\\ \text{countable}}} \sigma(X_t\colon t \in S)$$

Problem 5: Use the previous exercise to show that, given an $\mathbb{R}$-valued stochastic process $\{X_t\colon t \in [0,\infty)\}$ realized via coordinate projections on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes[0,\infty)})$, we have

$$\big\{t \mapsto X_t \text{ is continuous}\big\} \notin \sigma\big(X_t\colon t \in [0,\infty)\big)$$

Show also that for any $A \in \sigma(X_t\colon t \in [0,\infty))$,

$$A \subseteq \big\{t \mapsto X_t \text{ is continuous}\big\} \ \Rightarrow\ A = \emptyset$$

and thus $\{t \mapsto X_t \text{ is continuous}\}$ is not equivalent to a non-trivial measurable set under any probability measure P on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes[0,\infty)})$.