HW#1: due Fri 1/19/2024

This problem sets starts pretty lightly by a collection of problems on the homogeneous Poisson process and standard Brownian motion defined in the first lecture. The last two problems are a standard exercises in measure theory.

Problem 1: Consider the homogeneous Poisson process $\{N_t : t \geq 0\}$ as defined in class. Prove that, with probability one, all jumps of N are by $+1$. *Hint:* First prove that N is non-decreasing a.s. and then discretize time suitably. Note how (as it should) right-continuity enters your argument.

Problem 2: ØKSENDAL EX 2.8, PAGE 17

Problem 3: Prove that conditions 2-3 in our definition of standard Brownian motion — namely, the independence of increments along with $B_t - B_s = \mathcal{N}(0, |t-s|)$ — are equivalent to

$$\{B_t : t \geq 0\} \text{ is Gaussian with } \forall t, s \geq 0\colon\ EB_t = 0 \ \wedge\ E(B_t B_s) = t \wedge s$$

Here we recall that a stochastic process $\{X_t : t \in T\}$ is said to be Gaussian (a.k.a. multivariate normal) if for any $\lambda\colon T \to \mathbb{R}$ with $\{t \in T : \lambda(t) \neq 0\}$ finite, the (effectively finite) sum $\sum_{t\in T} \lambda(t) X_t$ is (univariate) normal.

Problem 4: A d-dimensional Brownian motion $\{B_t : t \geq 0\}$ is an $\mathbb{R}^d$-valued process whose Cartesian components are independent standard Brownian motions. Let

$$L(A) = \int_0^\infty 1_{\{B_s \in A\}} \mathrm{d}s$$

be the total time spent by the process in a Borel set A. (Check that this is well defined.) Prove that if A is a Lebesgue null set, then $L(A) = 0$ a.s. (This is Øksendal ex. 2.14 page 19. The random object L is the *occupation time* measure.)

Problem 5: Let $\{B_t : t \geq 0\}$ be the d-dimensional Brownian motion and let U be an orthogonal matrix. Show that also UB_t is a d-dimensional Brownian motion. (This is Øksendal ex. 2.15 page 19. The process is thus rotationally invariant.)

Problem 6: As a review, prove the (what the textbook calls) Doob-Dynkin lemma: If X is an $\mathbb{R}$-valued random variable and Y is $\sigma(X)$-measurable, then there exists a Borel measurable $f\colon \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.

Problem 7: Prove that if $\mathscr{X}$ is a complete and separable metric space endowed with the σ-algebra Σ of its Borel sets, then every probability measure μ on $(\mathscr{X}, \Sigma)$ is inner regular in the sense

$$\forall A \in \Sigma\colon \quad \mu(A) = \sup\{\mu(C) : C \subseteq A \text{ compact}\}$$

It may help to observe first that if we extend the infimum to closed sets, then this is true with no assumptions on the metric/topology of $\mathscr{X}$.