Problem 1: Let \( f \in C(\mathbb{R}) \) be such that, for some \( g, h \in C(\mathbb{R}) \) and with \( \{ B_t : t \geq 0 \} \) denoting the standard Brownian motion started from zero,
\[
f(B_t) = f(B_0) + \int_0^t g(B_s) \, dB_s + \frac{1}{2} \int_0^t h(B_s) \, ds
\]
holds a.s. for each \( t \geq 0 \). Prove that \( f \in C^2(\mathbb{R}) \) and \( g = f' \) and \( h = f'' \).

Problem 2: Let \( M_t \) be a local Itô martingale; i.e., \( M_t = \int_0^t Y_s \, dB_s \) for a jointly-measurable process \( Y \) adapted to a Brownian filtration, defined via localization for all \( t < \tau_{\infty} \) where \( \tau_{\infty} := \inf \{ t \geq 0 : \int_0^t Y_s^2 \, ds = \infty \} \)
Assuming that \( \int_0^\infty Y_s^2 \, ds = \infty \) a.s., prove that
\[
\limsup_{t \uparrow \tau_{\infty}} M_t = +\infty \quad \text{and} \quad \liminf_{t \uparrow \tau_{\infty}} M_t = -\infty
\]
a.s. on \( \{ \tau_{\infty} < \infty \} \).

Problem 3: Let \( \{ B_t : t \geq 0 \} \) be standard Brownian motion started from zero defined on a probability space \( (\Omega, \mathcal{F}, P) \) and let \( \mathcal{F}_t^B := \sigma(B_s : s \leq t) \) be the associated filtration. Let \( a \in \mathbb{R} \). Prove that
\[
\forall t \geq 0 \forall A \in \mathcal{F}_t^B : \ Q(A) := \lim_{T \to \infty} P(A \mid B_T = a T)
\]
exists and show that \( B \) under \( Q \) has the law of \( \{ B_t + at : t \geq 0 \} \) under \( P \). (The conditioning on singular event “\( B_T = a T \)” is justified by noting that the conditional probability given \( \sigma(B_T) \) admits a unique continuous version once \( T > t \).)

Problem 4: Let \( h_n(x) := (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \) be the \( n \)-th Hermit polynomial normalized so that \( h_n(x) = x^n + p(x) \) where \( p(x) \) is a polynomial of degree at most \( n - 1 \). Prove that
\[
\forall n \geq 1 : \quad n! \int_0^1 \left( \int_0^{t_n} \left( \int_0^{t_2} dB_{t_1} \right) \ldots \right) dB_{t_n-1} dB_{t_n} = h_n(B_1)
\]
where the object on the left is the iterated Itô integral of function 1. \( \text{Hint:} \) Notice that \( \{ h_n \}_{n \geq 1} \) form an orthogonal basis in \( L^2(\mathbb{R}, B(\mathbb{R}), \mu) \) where \( \mu \) is the law of \( B_1 \).