Weak & Strong mixing

$(\mathcal{X}, \mathcal{G}, \mu, \varphi) = $ m.p.s., $\mu(\mathcal{X}) = 1$

- ergodicity
- weakly mixing

$$\forall A, B \in \mathcal{G}: \quad \frac{1}{n} \sum_{k=0}^{n-1} \left| \mu(\varphi^{-k}(A) \cap B) - \mu(A)\mu(B) \right| \xrightarrow[n \to \infty]{}$$

- strongly mixing

$$\forall A, B \in \mathcal{G}: \quad \mu(\varphi^{-n}(A) \cap B) \xrightarrow[n \to \infty]{} \mu(A)\mu(B)$$

- tail trivial

Lemma Let $\{X_k\}_{k\in\mathbb{Z}}$ be stationary. Set $\mathcal{T} = \bigcap_{n\geq 0} \sigma(X_n, X_{n+1}, \ldots) =$ (right) tail

($\varphi =$ left shift)

Then TFAE:

(1) $\mathcal{T}$ is trivial ($\forall A\in\mathcal{T}: \mu(A)\in\{0,1\}$)

(2) $\forall B\in\mathcal{F} := \sigma(X_k : k\geq 0)$ $\quad \sup_{A\in\mathcal{F}} |\mu(\varphi^{-n}(A)\cap B) - \mu(A)\mu(B)| \xrightarrow[n\to\infty]{} 0$

Pf: $\boxed{(1)\Rightarrow(2)}$ $\quad |\mu(\varphi^{-n}(A)\cap B) - \mu(A)\mu(B)| = |E(\underbrace{1_{\varphi^{-n}(A)}}_{\in \mathcal{T}_n := \sigma(X_k : k\geq n)} (1_B - \mu(B)))|$

$= |E(1_{\varphi^{-n}(A)}[E(1_B|\mathcal{T}_n) - \mu(B)])| \leq E(|E(1_B|\mathcal{T}_n) - \mu(B)|)$

Since $\mathcal{T}_n \downarrow \mathcal{T}$, Backward Lévy thm:

$$E(1_B|\mathcal{T}_n) \xrightarrow[n\to\infty]{a.s.} E(1_B|\mathcal{T}) \overset{\mathcal{T}\text{-trivial}}{=} \mu(B)$$

Now apply BCT.

($\varphi^{-1} :=$ right shift)

$\boxed{(2)\Rightarrow(1)}$ (2) gives uniform convergence in $A\in\mathcal{F}$. Choose: $A := \varphi^{n}(B) \overset{\forall n\geq 1}{\in} \mathcal{F}$ if $B\in\mathcal{T}$.

to get $\mu(\varphi^{-n}(\varphi^{n}(B))\cap B) = \mu(B) \longrightarrow \mu(B)^2$ so $\mu(B) = \mu(B)^2 \Rightarrow \mu(B)\in\{0,1\}$.

$\boxtimes$

Lemma Let $\{X_k\}_{k\in\mathbb{Z}}$ be Bernoulli, $Z_n := \sum_{m\geq 0} 2^{-m} X_{n-m}$. Then $\{Z_k\}_{k\geq 0}$ are strongly mixing but $\bigcap_{n\geq 0} \sigma(Z_n, Z_{n+1}, \ldots) = \sigma(X_k : k\in\mathbb{Z})$; so NOT tail trivial.

Pf: Strong mixing: ETS for events of the form $A := \bigcap_{k=0}^{N} \{Z_k \geq l_k 2^{-n}\} \in \sigma(X_k : k \geq -n)$, $l_k \in \mathbb{N}$.

So $\varphi^{-m}(A) \in \sigma(X_k : k \geq m-n)$, if A' is another such event, then $\varphi^{-m}(A) \perp\!\!\!\perp A'$ if $m-n > N$.

This gives strong mixing.

On the other hand: $\sigma(Z_n) = \sigma(X_k : k \leq n)$ and so $\bigcap_{n\geq 0} \sigma(Z_n, Z_{n+1}, \ldots) \supseteq \sigma(X_k : k\in\mathbb{Z})$.

So $\{Z\}$ not tail trivial.

(WATCH: Left-tail IS trivial)

Lemma Irrational rotations of the unit circle are ergodic but NOT strongly mixing

$\mathcal{X} := [0,1)$, μ = Lebesgue, $\varphi(x) := x+\alpha \bmod 1$, $\alpha \notin \mathbb{Q}$

Pf: Weil Equidistribution theorem $\Rightarrow$ ergodicity

This implies $\{n\alpha \bmod 1 : n \geq 0\}$ is dense in $[0,1)$.

Therefore, $\exists\, n_k \to \infty$ s.t. $n_k\alpha \bmod 1 \longrightarrow 1/2$.

Take $A = [0, 1/4]$. Then $\varphi^{-n_k}(A) \subseteq [1/2, 2/3]$ for k large.

So $\varphi^{-n_k}(A) \cap A = \emptyset$ for k large. So φ NOT strongly mixing.

WTS: Irrational rot. NOT weakly mixing.

Lemma (Koopman & von Neumann, 1932) Let $f: \mathbb{N} \to \mathbb{R}_+$ be bounded ($\forall n \geq 0: 0 \leq f(n) \leq M$).

TFAE: (1) $\frac{1}{n} \sum_{k=1}^{n} f(k) \longrightarrow 0$

(2) • $\exists E \subseteq \mathbb{N}$:

$$\lim_{n \to \infty} \frac{1}{n} \left| E \cap \{1, \dots, n\} \right| = 0 \quad \text{(zero density)}$$

• $\lim_{\substack{n \to \infty \\ n \notin E}} f(n) = 0$

Pf: $\boxed{(2) \Rightarrow (1)}$ Pick $\varepsilon > 0$. Then $\exists n_0 \geq 1$: $\forall n \in \{n_0, n_0+1, \dots\} \setminus E$: $|f(n)| < \varepsilon$.

Then $0 \leq \frac{1}{n} \sum_{k=1}^{n} f(k) \leq \frac{M n_0}{n} + \frac{|E \cap \{1, \dots, n\}|}{n} M + \varepsilon \xrightarrow[n \to \infty]{} \varepsilon$.

$\boxed{(1) \Rightarrow (2)}$ For $m \geq 1$, set $E_m := \{ k \geq 1 : f(k) > \frac{1}{m} \}$. Note $E_m \uparrow$.

Also $\frac{1}{n} \sum_{k=1}^{n} 1_{E_m}(k) \leq \frac{1}{n} \sum_{k=1}^{n} m f(k) \xrightarrow[n \to \infty]{} 0$ so E_m zero density.

Now define $i_0 = 0 < i_1 < i_2 < \dots$ recursively so that

$$\forall n \geq i_{m-1}: \frac{1}{n} \sum_{k=1}^{n} 1_{E_m}(k) < \frac{1}{m} \quad (m \geq 1)$$

Define $E := \bigcup_{m \ge 1} E_m \cap \{i_{m-1}, \dots, i_m\}$. $\quad$ (because $E_k \uparrow$)

Then for n s.t. $i_{m-1} \le n < i_m$: $\quad E \cap \{1, \dots, n\} \subseteq E_m$

so $\frac{1}{n} \sum_{k=1}^{n} 1_E(k) \le \frac{1}{n} \sum_{k=1}^{n} 1_{E_m}(k) < \frac{1}{m} \xrightarrow[n \to \infty]{} 0$ so E is zero density

For same n s.t. $n \notin E \Rightarrow n \notin E_m$ and so $f(n) \le \frac{1}{m}$,

and thus $\sup\{f(k) : k \ge i_{m-1}, k \notin E\} \le \frac{1}{m}$

hence $\lim_{\substack{n \to \infty \\ n \notin E}} f(n) = 0$. $\boxtimes$

<u>Def</u> For $f : \mathbb{N} \to \mathbb{R}$ bounded we say

$$D\text{-}\lim_{n \to \infty} f(n) = c \qquad \text{"limit in density"}$$

if above holds for $|f(\cdot) - c|$, or equivalently, if $\frac{1}{n} \sum_{k=1}^{n} |f(k) - c| \longrightarrow 0$.

<u>Note</u> $\frac{1}{n} \sum_{k=1}^{n} |f(k) - c|^2 \longrightarrow 0$ also $\iff$ $D\text{-}\lim_{n \to \infty} f(n) = c$.