Recurrence in ∞-measure spaces

$$\cdots \to \varphi^{-2}(W) \xrightarrow{\varphi} \varphi^{-1}(W) \xrightarrow{\varphi} W$$

<u>Def</u> $W \in \mathcal{G}$ <u>wandering</u> if $W \cap \bigcup_{k\geq 1} \varphi^{-k}(W) = \emptyset$

or, equivalently, if $\{\varphi^{-k}(W)\}_{k\geq 0}$ are disjoint.

<u>Thm</u> (Halmos 1947) Let $(\mathcal{X}, \mathcal{G}, \mu)$ = meas. space, $\varphi : \mathcal{X} \to \mathcal{X}$ meas. map. Then TFAE:

(1) φ is <u>recurrent</u> meaning $\forall A \in \mathcal{G} : \mu\left(A \setminus \bigcup_{k\geq 1} \varphi^{-k}(A)\right) = 0$

(2) φ is <u>conservative</u> —〃— $\forall A \in \mathcal{G} : A$ wandering $\Rightarrow \mu(A) = 0$

(3) φ is <u>incompressible</u> —〃— $\forall A \in \mathcal{G} : \varphi^{-1}(A) \subseteq A \Rightarrow \mu(A \setminus \varphi^{-1}(A)) = 0$

If φ^{-1} preserves null-sets $\left(\forall A \in \mathcal{G} : \mu(A) = 0 \Rightarrow \mu(\varphi^{-1}(A)) = 0\right)$ then also

(4) φ is <u>infinitely recurrent</u> —〃— $\forall A \in \mathcal{G} : \mu\left(A \setminus \bigcap_{n\geq 0} \bigcup_{k\geq n} \varphi^{-k}(A)\right) = 0$

Pf: $\underline{(1) \Leftrightarrow (2)}$ $\quad W := A \setminus \bigcup_{k \ge 1} \varphi^{-k}(A)$ is wandering

if A is wandering, then $W = A$.

$\underline{(1) \Leftrightarrow (3)}$ $\quad B := A \cup \bigcup_{k \ge 1} \varphi^{-k}(A).$

$(3) \Rightarrow (1)$ Note $\varphi^{-1}(B) \subseteq B$ and $B \setminus \varphi^{-1}(B) = A \setminus \bigcup_{k \ge 1} \varphi^{-k}(A)$

$(1) \Rightarrow (3)$ if $\varphi^{-1}(A) \subseteq A \Rightarrow A \setminus \varphi^{-1}(A) = A \setminus \bigcup_{k \ge 1} \varphi^{-k}(A)$

$\underline{(1) \Leftrightarrow (4)}$ $\quad (4) \Rightarrow (1)$ immediate

$(1) \Rightarrow (4)$:

$$\left(A \setminus \bigcap_{n \ge 0} \bigcup_{k \ge n} \varphi^{-k}(A)\right) \setminus \left(A \setminus \bigcup_{k \ge 1} \varphi^{-k}(A)\right)$$

$$\subseteq \bigcup_{n \ge 0} \left[\varphi^{-n}(A) \setminus \bigcap_{k \ge n+1} \varphi^{-k}(A)\right]$$

$$= \bigcup_{n \ge 0} \varphi^{-n}\left(A \setminus \bigcup_{k \ge 1} \varphi^{-k}(A)\right) \overset{(1)}{=} \text{null set}$$

Thm (Hopf decomposition) Let $(\mathcal{X}, \mathcal{G}, \mu)$ be meas. space, μ - σ finite. Let $\varphi : \mathcal{X} \to \mathcal{X}$ be bimeasurable bijection, s.t. φ, φ^{-1} preserve null sets. Then $\exists\, W \in \mathcal{G}$ s.t.

- W is wandering
- the sets $D := \bigcup_{n \in \mathbb{Z}} \varphi^n(W)$

  $C := \mathcal{X} \setminus D$

  are preserved by φ. $\left(\begin{array}{l} \varphi(C) = C \\ \varphi(D) = D \end{array} \right)$

Moreover, $\varphi|_C$ is conservative.

$g, \mu)$

<u>Lemma</u>: $\forall W \in \mathcal{G}$: W wandering $\Rightarrow \{\varphi^n(W)\}_{n\in\mathbb{Z}}$ disjoint

Moreover, if $W, W' \in \mathcal{G}$ are wandering sets then $\exists \tilde{W} \in \mathcal{G}$ wandering s.t.

$$W \cup \tilde{W} \text{ wandering} \wedge \bigcup_{k\in\mathbb{Z}} \varphi^k(W\cup\tilde{W}) = \bigcup_{k\in\mathbb{Z}} \varphi^k(W) \cup \bigcup_{k\in\mathbb{Z}} \varphi^k(W')$$

Pf: Set $\tilde{W} := W' \setminus \bigcup_{k\in\mathbb{Z}} \varphi^k(W)$. Wandering as $\tilde{W} \subseteq W'$.

Also $\tilde{W} \cap \bigcup_{k\in\mathbb{Z}} \varphi^k(W) = \emptyset \wedge W \cap \bigcup_{k\in\mathbb{Z}} \varphi^k(\tilde{W}) = \emptyset$

Hence

$$\begin{aligned}(W\cup\tilde{W}) \cap \varphi^k(W\cup\tilde{W}) &= W\cap\varphi^k(W) \cup \tilde{W}\cap\varphi^k(\tilde{W}) \\ &\cup W\cap\varphi^k(\tilde{W}) \cup \tilde{W}\cap\varphi^k(W) \overset{k\neq 0}{=} \emptyset\end{aligned}$$

So $W\cup\tilde{W}$ is wandering.

Now

$$\bigcup_{n\in\mathbb{Z}} \varphi^n(W\cup\tilde{W}) = \bigcup_{n\in\mathbb{Z}} \varphi^n(W) \cup \bigcup_{n\in\mathbb{Z}} \varphi^n(\tilde{W}) = \bigcup_{n\in\mathbb{Z}} \varphi^n(W) \cup \left[\bigcup_{k\in\mathbb{Z}} \varphi^n(W') \setminus \bigcup_{n\in\mathbb{Z}} \varphi^n(W)\right]$$

Pf of Thm: μ-σ-finite $\Rightarrow \exists \{\mathcal{X}_n\}_{n\geq 0}$ disjoint, $\mathcal{X} = \bigcup_{n\geq 0} \mathcal{X}_n$

$\forall n \geq 0: 0 < \mu(\mathcal{X}_n) < \infty.$

Consider

$$\sup\left\{ \sum_{n\geq 0} 2^{-n} \frac{1}{\mu(\mathcal{X}_n)} \mu\left(\mathcal{X}_n \cap \bigcup_{k\in\mathbb{Z}} \varphi^k(W)\right) : W \in \mathcal{G} \text{ wandering} \right\}.$$

If $\{W_k\}$ is maximizing sequence $\xRightarrow{\text{Lemma}}$ may assume $W_k \uparrow$.

Denote: $W := \bigcup_{k\geq 1} W_k$.

Claim: W is wandering:

$$W \cap \varphi^k(W) = \bigcup_{j,l\geq 1} W_j \cap \varphi^k(W_l) \subseteq \bigcup_{j\geq 1} W_j \cap \varphi^k(W_j) \underset{k\neq 0}{=} \emptyset$$

MCT: W is maximizer.

Lemma: $\forall W'$ wandering, $W' \cap \bigcup_{k=0} \varphi^k(W) = \emptyset$

$\Rightarrow \mu(W') = 0.$ $\boxtimes$

Thm (Hopf Ratio Ergodic Thm, Stephanov '36, Hopf '37)

Let $(\mathcal{X}, \mathcal{B}, \mu, \varphi)$ = m.p.s., $f, g \in L^1(\mu)$ s.t. $g \geq 0$.

Then $\exists\, R \in L^0(\mathcal{X}, \mathcal{B}, \mu)$ s.t. $R \circ \varphi = R$ a.e. s.t.

$$\frac{\sum_{k=0}^{n-1} f \circ \varphi^k}{\sum_{k=0}^{n-1} g \circ \varphi^k} \longrightarrow R \quad \text{a.e. on } \left\{ \sum_{k=0}^{\infty} g \circ \varphi^k \cancel{> 0} = \infty \right\}.$$

Moreover, $Rg \in L^1$ and $\int f \, d\mu = \int R g \, d\mu$.

Pf by Kamae & Keane:

Lemma Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ be ≥ 0 sequences and $M \geq 1$ s.t.

$$\forall n \geq 0 \ \exists m = 1 \ldots M: \ \sum_{i=0}^{m-1} a_{i+n} \geq \sum_{i=0}^{m-1} b_{i+n}$$

Then $\forall N > M$: $\displaystyle\sum_{i=0}^{N-1} a_i \geq \sum_{i=0}^{N-M-1} b_i$

Pf Assumption gives $m_0 = 0 < m_1 < \ldots < m_k < N$ s.t.

$$\forall i = 1 \ldots k: \ m_i - m_{i-1} \leq M \ \wedge \sum_{m_{i-1} \leq j < m_i} a_j \geq \sum_{m_{i-1} \leq j < m_i} b_j$$

Summing the blocks gives the claim.

$$\sum_{j=0}^{N-1} a_j \geq \sum_{i=1}^{k} \sum_{m_{i-1} \leq j < m_i} a_j$$

$$\geq \sum_{i=1}^{k} \sum_{m_{j-1} \leq j < m_j} b_j$$

$$= \sum_{j=0}^{m_k-1} b_j \geq \sum_{j=0}^{N-K-1} b_j \qquad \boxtimes$$

Pf of Thm next time: