Recurrence

Thm (Poincaré 1899) Let $(\mathfrak{X}, \mathcal{G}, \mu, \varphi) =$ m.p.s., $\mu(\mathfrak{X}) < \infty$.

Then $\forall A \in \mathcal{G}: \quad \mu\Big(A \setminus \bigcup_{k \geq 1} \varphi^{-k}(A)\Big) = 0$

$\{x \in A : (\forall k \geq 1 : \varphi^k(x) \notin A)\}$.

Set $n_A(x) := \inf\{n \geq 1 : \varphi^n(x) \in A\}$ first return time to A

$\boxed{\textbf{Thm:} \quad n_A < \infty \text{ a.e. on } A}$

note $\varphi^{-1}(W) \cap W = \emptyset$

Remark Fails when $\mu(\mathfrak{X}) = \infty$. Ex $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathrm{Leb})$, $\varphi(x) := x + 1$

$A := [0,1)$

Q: How long does it take to return?

Thm (Khinchine, 1934) Let $(\mathcal{X}, \mathcal{G}, \mu, \varphi)$ = m.p.s., $\mu(\mathcal{X}) = 1$.
Let $A \in \mathcal{G}$ and $\varepsilon \in (0, \mu(A)^2)$. Then

$$E_\varepsilon := \{ n \geq 0 : \mu(\varphi^{-n}(A) \cap A) \geq \mu(A)^2 - \varepsilon \}$$

has bounded gaps:

$$\exists n \geq 1 \ \forall k \geq 0 : E_\varepsilon \cap \{k, k+1, \ldots, k+n-1\} \neq \emptyset$$

Pf: By Hilbert space calculus: $Tf := f \circ \varphi$.

$$\mu(\varphi^{-n}(A) \cap A) = \langle T^n 1_A, 1_A \rangle$$

$\mu(X)=1$:

$-\varepsilon \}$

ϕ

Recall Mean Erg. Thm: $\frac{1}{n}\sum_{k=0}^{n-1} T^k f \xrightarrow[n\to\infty]{L^2} Pf$

$P = \mathrm{proj}_{\ker(1-T)}$

So given $\varepsilon>0$, $\exists n\geq 1$:

$$\left\| \frac{1}{n}\sum_{i=0}^{n-1} T^i 1_A - P1_A \right\| < \frac{\varepsilon}{1+\|1_A\|}$$

T - unitary so:

$$\forall k\geq 0: \left\| \frac{1}{n}\sum_{i=k}^{n+k-1} T^i 1_A - P1_A \right\| < \frac{\varepsilon}{1+\|1_A\|}$$

So

$$\left| \left\langle \frac{1}{n}\sum_{i=k}^{n+k-1} T^i 1_A - P1_A , 1_A \right\rangle \right|$$

$$\leq \| \ldots \| \, \|1_A\|$$

$$< \frac{\varepsilon}{1+\|1_A\|}\|1_A\| \leq \varepsilon .$$

This

$\frac{1}{n}$

Now

$<$

So

Q:

$1_A\|$

$1_A\|$

This gives

$$\frac{1}{n}\sum_{i=k}^{n+k-1} \mu(\varphi^{-i}(A)\cap A) = \left\langle \frac{1}{n}\sum_{i=k}^{n+k-1} T^i 1_A,\, 1_A \right\rangle$$

$$\geq \langle P1_A, 1_A\rangle - \varepsilon \;\geq\; \mu(A)^2 - \varepsilon .$$

Now

$$\langle P1_A, 1_A\rangle = \langle P1_A, P1_A\rangle$$

$$\geq \frac{\langle P1_A, 1\rangle^2}{\langle 1,1\rangle} \underset{P1=1}{=} \frac{\langle 1_A, 1\rangle^2}{\langle 1,1\rangle} \underset{\mu(Z)=1}{=} \mu(A)^2$$

So $E_\varepsilon \cap \{k, k+1, \ldots, k+n-1\} \neq \emptyset$: $\boxtimes$

Q: Return time for point?

Thm (Kac 1947) Let $(\mathcal{X}, \mathcal{G}, \mu, \varphi) =$ m.p.s., $\mu(\mathcal{X}) < \infty$.

Then $\forall A \in \mathcal{G}$: $\int_A n_A \, d\mu \leq \mu(\mathcal{X})$

If φ - ergodic and $\mu(A) > 0$ then equality holds.

Pf Set $A_n := A \cap \{n_A = n\}$.

Then $\{A_n\}_{n \geq 1}$ disjoint

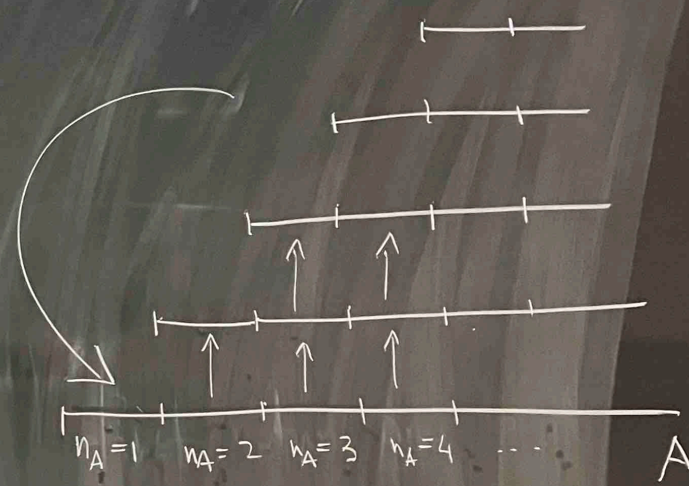
Next $j = 1, \dots, n-1$:

$$\varphi^{-j}(A_n) \cap A_n = \{x \in A : n_A = n, \varphi^{j}(x) \in A\} = \emptyset$$

Hence, $\{\varphi^{-j}(A_n) : 0 \leq j < n, n \geq 1\}$ is disjoint family.

Also: $\forall j = 0, \dots, n-1$: $\mu(\varphi^{-j}(A_n)) = \mu(A_n)$

Kakutani skyscraper



Now:

$$\int_A n_A \, d\mu \overset{\text{Poincaré}}{=} \sum_{n\geq 1} n\,\mu(A_n)$$

$$= \sum_{n\geq 1} \sum_{k=0}^{n-1} \mu(\varphi^{-k}(A_n))$$

$$\overset{\text{disj.}}{=} \mu\left(\bigcup_{n\geq 1} \bigcup_{k=0}^{n-1} \varphi^{-k}(A_n)\right) \leq \mu(\mathcal{X}).$$

Note $\varphi^{-n}(A_n) \subseteq A \underset{\text{a.e.}}{\subseteq} \bigcup_{n\geq 1} \bigcup_{k=0} \varphi^{-k}(A_n)$

So $\varphi^{-1}\left(\bigcup_{n\geq 1} \bigcup_{k=0}^{n-1} \varphi^{-k}(A_n)\right) = \bigcup_{n\geq 1} \bigcup_{k=1}^{n} \varphi^{-k}(A_n)$

$$\underset{\text{a.e.}}{\subseteq} \bigcup_{n\geq 1} \bigcup_{k=0}^{n-1} \varphi^{-k}(A_n)$$

Since φ is m.p.t., $\Rightarrow$ $\overset{\text{a.e.}}{=}$ holds.

i.e. $S_A := \bigcup_{n \geq 1} \bigcup_{k=0}^{n-1} \varphi^{-k}(A_n) \in \mathcal{B}'$ (almost invariant)

Since $\mu(A) > 0$, $\mu(S_A) > 0$.

Under ergodicity:

$$\int_A n_A \, d\mu = \mu(S_A) = \mu(\mathcal{H}). \quad \boxtimes$$

Kac thm: ($\mu(A) > 0$)

$$\frac{1}{\mu(A)} \int_A n_A \, d\mu \leq \left(\frac{\mu(A)}{\mu(\mathcal{H})} \right)^{-1}$$

Some consequences: (of Poincaré)

$\mu(\mathcal{X})<\infty$, $\mu(A)>0$: Define $\varphi_A : A \to A$

$$\varphi_A(x) := \begin{cases} \varphi^{n_A(x)}(x) & \text{if } n_A(x)<\infty \\ x & n_A(x)=\infty \end{cases}$$

(Induced / derivative map)

Lemma Set $\mathcal{G}_A := \{B\cap A : B\in\mathcal{G}\}$

$\mu_A(B) := \mu(A\cap B)$.

Then (1) φ is $\mathcal{G}$-meas $\Rightarrow$ φ_A is $\mathcal{G}_A$-meas

(2) φ is m.p.t. on $(\mathcal{X},\mathcal{G},\mu)$ $\Rightarrow$ φ_A m.p.t. on $(A,\mathcal{G}_A,\mu_A)$

(3) φ is ergodic w.r.t. μ $\Rightarrow$ φ_A ergodic w.r.t. μ_A.

Ex $\{X_k\}_{k\in\mathbb{Z}}$ stationary, 0-1-valued, $A=\{X_0=0\}$, φ = left shift.

$\Rightarrow$ φ_A = shift to next zero.

Recurrence in infinite measure space

<u>Def</u>: A set $W \in \mathcal{G}$ is <u>wandering</u> if

$$W \cap \bigcup_{k \geq 1} \varphi^{-k}(W) = \emptyset$$

equivalently: $\{\varphi^{-k}(W)\}_{k \geq 0}$ are pairwise disjoint