From semigroup to Markov process

Last time: constructed $\{\hat{X}_t : t \in \mathbb{Q}^+\}$ with "correct" distribution.

Next step: Extract càdlàg-version of this process.

Key idea:

$$e^{-\lambda t} T(t) \overbrace{R(\lambda) g}^{f} = \int_t^\infty e^{-\lambda s} T(s) g \underset{g \geq 0}{\leq} \int_0^\infty e^{-\lambda s} T(s) g = R(\lambda) g.$$

(by pos. preserving)

For f we then get

$$e^{-\lambda(t+s)} T(t+s) f \leq e^{-\lambda s} T(s) f$$

Rewrite using $\hat{X}$:

$$E\left(e^{-\lambda(t+s)} f(\hat{X}_{t+s}) \,\middle|\, \mathcal{F}_s\right) \leq e^{-\lambda s} f(\hat{X}_s)$$

$\mathcal{F}_s := \sigma(\hat{X}_u : u \leq s)$

So $\{ e^{-\lambda t} f(\tilde{X}_t) : t \geq 0 \}$ is ≥ 0 supermartingale.

<u>Now:</u> Let M_t be ≥ 0 supermartingale.

Then

$$P\Big(\forall t \geq 0 : \lim_{\substack{s \downarrow t \\ (s \in \mathbb{Q})}} M_s \text{ exists} \;\wedge\; \forall t > 0 : \lim_{\substack{s \uparrow t \\ (s \in \mathbb{Q})}} M_s \text{ exists} \Big) = 1.$$

and

$$\forall t \geq 0 : P(M_t > 0) = 1$$

implies

$$P\big(\forall t \geq 0 : M_t > 0 \big) = 1$$

Using test

$\forall x \in S$:

$P^x\big(\forall t \geq 0$:

$\forall t$

Now define

X_t :

and conclude

Using test functions we now conclude

$\forall x \in S$:

$$P^x\Big(\forall t \geq 0: \lim_{\substack{s \downarrow t \\ s \in \mathbb{Q}}} \widehat{X}_s \text{ exists in } S' \quad \forall t > 0: \lim_{\substack{s \uparrow t \\ s \in \mathbb{Q}}} \widehat{X}_s \text{ exists in } S'\Big) = 1$$

Now define

$$X_t := \lim_{\substack{s \downarrow t \\ s \in \mathbb{Q}}} \widehat{X}_s$$

and conclude $t \longmapsto X_t$ càdlàg a.s.

Ex

Def

Then

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Thm L

(1)

(2)

(3)

Ex Standard Brownian motion:

Def Let $S = \mathbb{R}$, $T(t)f(x) = \mathbb{E}\left(f(x+N(0,t))\right)$

Then $\{T(t)\}$ is prob. semigroup and the associated M.P. is called standard Brownian motion

SBM

Thm Let X = SBM. Then

(1) $\forall n \geq 1$ $\forall\, 0 < t_0 < t_1 \ldots < t_n$:

$\{X_{t_i} - X_{t_{i-1}}\}_{i=1}^{n}$ are independent.

(2) $\forall\, 0 \leq s < t$: $X_t - X_s = N(0, t-s)$

(3) $t \longmapsto X_t$ is a.s. continuous.

Pf: MP definition: $\forall\, s,t \geq 0$:

$$E^x\left(f(X_{t+s}) \mid \mathcal{F}_s\right) \overset{MP}{=} E^{X_s}\left(f(X_t)\right)$$

$$= T(t)f(X_s) = \mathbb{E}\left(f(\cdot + N(0,t)\right)(X_s)$$

Now apply $(\forall)$ to $f(x) = e^{i\lambda x}$:

$$E^x\left(e^{i\lambda X_{t+s}} \mid \mathcal{F}_s\right) = e^{i\lambda X_s}\, \underbrace{\mathbb{E}\left(e^{i\lambda N(0,t)}\right)}_{e^{-\frac{\lambda^2}{2}t}}$$

So

$\forall s \leq t$: $$E^x\left(e^{i\lambda(X_t - X_s)} \mid \mathcal{F}_s\right) = e^{-\frac{\lambda^2}{2}(t-s)}$$

This gives

$$E^x\left(e^{i\sum_{j=1}^{n} \lambda_j (X_{t_j} - X_{t_{j-1}})}\right) = e^{-\frac{1}{2}\sum_{j=1}^{n} \lambda_j^2 (t_j - t_{j-1})}$$

proving (1-2). For (3), need Kolmogorov-Censtov Thm (275) $\boxtimes$

Analogue of Q-matrix $\longrightarrow$ generator

Def Given any family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on Banach space $\mathcal{B}$, define

$$\mathrm{Dom}(\mathcal{L}) := \left\{ f \in \mathcal{B} : \lim_{t \downarrow 0} \frac{T(t)f - f}{t} \text{ exists} \right\}$$

linear space

and for $f \in \mathrm{Dom}(\mathcal{L})$ set

$$\mathcal{L}f := \lim_{t \downarrow 0} \frac{T(t)f - f}{t}$$

Note For conti. semigroups this will be densely defined, linear, (unbounded).

Thm Let $\{T(t)\}_{t\geq 0}$ be strongly continuous semigroup of contractions on Banach space $\mathcal{B}$.

Then:

(1) $\mathrm{Dom}(\mathcal{L})$ is dense in $\mathcal{B}$

(2) $\forall \lambda > 0$: $\mathrm{Ran}(1 - \lambda^{-1}\mathcal{L}) = \mathcal{B}$

(3) $\forall \lambda > 0$: $\|(1-\lambda^{-1}\mathcal{L})^{-1}\| \leq 1$

(4) $\forall t > 0\ \forall f \in \mathcal{B}$: $T(t)f = \lim_{n\to\infty} \left(1 - \frac{t}{n}\mathcal{L}\right)^{-n} f$

(5) $\forall t > 0\ \forall f \in \mathrm{Dom}(\mathcal{L})$:

$$\frac{d}{dt} T(t)f = T(t)\mathcal{L}f = \mathcal{L}T(t)f$$

and $T(t)f \in \mathrm{Dom}(\mathcal{L})$

Pf

Pf Key tool: resolvent

$$R(\lambda)f := \int_0^\infty e^{-\lambda t} T(t) f \, dt$$

(defined as improper Riemann integral)

Note Set $f := \lambda R(\lambda) g$.

Then

$$T(t)f - f = \lambda e^{\lambda t} \int_t^\infty e^{-\lambda s} T(s) g$$

$$- \lambda \int_0^\infty e^{-\lambda s} T(s) g$$

$$= (e^{\lambda t} - 1) f - \lambda e^{\lambda t} \int_0^t e^{-\lambda s} T(s) g \, ds$$

So $\dfrac{T(t)f - f}{t} \xrightarrow[t \downarrow 0]{} \lambda f - \lambda g$

So f

proving

Fact to

Then we

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For (5):

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So $f = \lambda R(\lambda) g \Rightarrow f \in \mathrm{Dom}(\mathcal{L}) \wedge g = f - \lambda^{-1}\mathcal{L} f$

proving (1) and (2).

Fact to prove now: $\Longleftrightarrow$ holds here.

(by showing (4) for A)

Then we get

$$\|(1-\lambda^{-1}\mathcal{L})^{-1} g\| = \|f\| = \|\lambda R(\lambda) g\|$$

$$\leq \lambda \int_0^\infty e^{-\lambda t} \underbrace{\|T(t) g\|}_{\leq \|g\|} \, dt \leq \|g\|$$

So $\|(1-\lambda^{-1}\mathcal{L})^{-1}\| \leq 1$.

For (5):

$$(1-\lambda^{-1}\mathcal{L})^{-n} f = \lambda^n R(\lambda)^n f$$

$$= \int_{\mathbb{R}_+^n} dt_1 \cdots dt_n \, \lambda^n e^{-\lambda(t_1 + \cdots + t_n)} T(t_1 + \cdots + t_n) f$$

So $(1-\lambda^{-1}\mathcal{L})^{-n} f = \mathbb{E}\left(T\left(\frac{1}{\lambda}(Y_1 + \cdots + Y_n)\right) f\right)$

where $Y_1, \ldots, Y_n$ = iid Exp(1).

So $(1-\frac{t}{n}\mathcal{L})^{-n} f = \mathbb{E}\left(T\left(t \underbrace{\frac{Y_1 + \cdots + Y_n}{n}}_{\text{SLLN} \xrightarrow[n\to\infty]{} 1 \text{ a.s.}}\right) f\right)$

$\xrightarrow[n\to\infty]{} T(t)f.$ $\boxtimes$

<u>Note</u>: $e^{t\mathcal{L}}$ OR $(1+\frac{t}{n}\mathcal{L})^{n}$ doesn't make sense in general

$(1-\frac{t}{n}\mathcal{L})^{-n}$ does.