Ergodicity

φ m.p.t.

ET: $\displaystyle \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} f \circ \varphi_x \longrightarrow \bar{f}$ $\qquad f \in L^1$, $\bar{f} \in L^1$

$$\bar{f} = \bar{f} \circ \varphi \quad \text{a.e.}$$

Def: Let $(\mathcal{X}, \mathcal{G}, \mu, \varphi)$ = m.p.s. A set $A \in \mathcal{G}$ is

- **invariant** if $\varphi^{-1}(A) = A$
- **almost invariant** $\varphi^{-1}(A) = A$ a.e. $\quad \mu(A \,\triangle\, \varphi^{-1}(A)) = 0$.

Lemma

$\left\{ \begin{array}{l} \mathcal{I} := \{A \in \mathcal{G} : \text{invariant}\} \\ \mathcal{I}' := \{A \in \mathcal{G} : \text{almost inv.}\} \end{array} \right.$ are σ-algebras

- $\forall A' \in \mathcal{I}' \ \exists A \in \mathcal{I} : A' = A$ a.e. $\quad (\mu(A' \triangle A) = 0)$ $\quad (*)$
- $f = f \circ \varphi \iff f$ is $\mathcal{I}$-measurable

Pf: $A' \in \mathcal{I}'$: $A := \bigcap_{n\geq 1} \bigcup_{k\geq n} \varphi^{-k}(A')$

$= \{ x \in \mathcal{X} : \varphi^n(x) \in A' \text{ i.o.} \} \in \mathcal{I}$.

φ – meas. pres. $\quad \varphi^{-k}(A') = A'$ a.e.

$\Rightarrow \bigcup_{k\geq n} \varphi^{-k}(A') = A'$ a.e.

$\Rightarrow A = A'$ a.e.

Corollary (Birkhoff–Khinchin's ET)

Let $(\mathcal{X}, \mathcal{G}, \mu, \varphi)$ = m.p.s., / $\mu(\mathcal{X}) = 1$. Then

$\forall f \in L^1$: $\frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} f \circ \varphi_x \longrightarrow E(f \mid \mathcal{I})$ a.e.

$\text{i.o.}\} \in \mathcal{J}$.

Then

$\mathcal{J}) \quad$ a.s.

Pf Birkhoff ET: $\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} f\circ\varphi_x \to \bar{f}$ in L^1

Pick $A \in \mathcal{J}$. Then

$$E(f 1_A) \overset{\text{m.p.t.}}{=} E(f\circ\varphi_x\, 1_A\circ\varphi_x) \quad \forall x$$

$$\overset{A\in\mathcal{J}}{=} E(f\circ\varphi_x\, 1_A) \quad \forall x$$

$$= E\Big(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} f\circ\varphi_x\, 1_A\Big)$$

$$\longrightarrow E(\bar{f}\, 1_A)$$

where $\bar{f}$ is $\mathcal{J}$-meas. $\Rightarrow \bar{f} = E(f|\mathcal{J})$ a.s. $\boxtimes$

Def A m.p.t. φ on $(\mathcal{X}, \mathcal{G}, \mu)$ is *ergodic* if $\forall A\in\mathcal{J}$: $\mu(A)=0 \vee \mu(A^c)=0$

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Lemma φ-ergodic $\iff \forall h \in L^0 : h = h \circ \varphi \Rightarrow \exists c \in \mathbb{R} :$ $h = c$ a.e.

If $\mu(\mathfrak{X}) < \infty$, suffices to check $h \in L^\infty$.

Pf: $(\Rightarrow)$ Assume $h \in L^0$, $h = h \circ \varphi$.

Set $c := \sup\{ a \in \mathbb{R} : \mu(h < a) = 0 \}$

Since $\{h < c - 2^{-n}\} \uparrow \{h < c\}$ and so

by MCT: $\mu(h < c) = 0$.

Next: $\mu(h \leq c + 2^{-n}) > 0$ and since

$\{h \leq c + 2^{-n}\} \in \mathcal{I} \wedge \varphi$-erg. $\Rightarrow \mu(h > c + 2^{-n}) = 0$.

Now $\{h > c + 2^{-n}\} \uparrow \{h > c\} \Rightarrow \mu(h > c) = 0$.

$(\Leftarrow)$ φ NOT ergodic $\Rightarrow \exists A \in \mathcal{I} : \mu(A) > 0 \wedge \mu(A^c) > 0$.

Then $\mathbb{1}_A$ is φ-inv, but not $=$ const. a.e.

<u>Lemma</u> Let $(\mathcal{X}, \mathcal{G}, \mu, \varphi)$ = m.p.s. with $\mu(\mathcal{X}) < \infty$.

Then

$$\varphi\text{-ergodic} \iff \forall f \in L^1 : \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} f \circ \varphi_x \xrightarrow[\text{a.e.}]{} \frac{1}{\mu(\mathcal{X})} \int f \, d\mu$$

Pf Know convergence in L^1. So

$$\int \bar{f} \, d\mu = \int f \, d\mu$$

φ-ergodic $\iff \forall f \in L^1$: $\bar{f}$ = constant a.e. $\boxtimes$

Examples:

"Bernoulli shifts"

Lemma Shifts of iid sequences are ergodic

Pf: $\{X_k\}_{k\geq 0}$ iid, S-valued

$$\left.\begin{array}{l}(S^{\mathbb{N}}, \Sigma^{\otimes\mathbb{N}}, \nu^{\otimes\mathbb{N}}, \theta) \\ \theta := \text{left shift}\end{array}\right\} \text{ m.p.s.}$$

$E \in \mathcal{I}$. $\theta^{-1}(E) \in \sigma(X_1, X_2, \dots)$

$\theta^{-n}(E) \in \sigma(X_n, X_{n+1}, \dots)$

So $E = \theta^{-1}(E) \Rightarrow E$ is tail event!

Kolmogorov 0-1 law $\Rightarrow \nu^{\otimes\mathbb{N}}(E) = \begin{cases} 0 \\ 1 \end{cases}$ $\blacksquare$

Corollary "Continued fractions" m.p.s. and Baker's transform are ergodic

"Pf" $x \overset{\in [0,1]\setminus\mathbb{Q}}{\longleftrightarrow} \{a_k\}_{k\geq 1}$ by $x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$

$\mu(dx) = \frac{1}{\mathcal{Z}}\chi^{(x)} \frac{1}{1+x}\,dx \longleftrightarrow \{a_k\}_{k\geq 1}$ iid

Baker's transform similar.

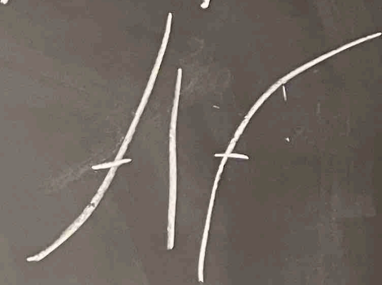
Lemma Irrational rotations of circle are ergodic.

Pf $\mathcal{X} = [0,1)$, $\varphi_\alpha(x) = x + \alpha \mod 1$, Leb.

Weil equidistribution thm:

$$\alpha \notin \mathbb{Q} \Rightarrow \forall f \in L^1 : \frac{1}{n}\sum_{k=0}^{n-1} f\circ\varphi_\alpha^k \xrightarrow[\text{a.e.}]{} \int_0^1 f\,dx$$

Thm (Boole transform) On $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \text{Leb})$



Set $\varphi(x) := \begin{cases} x - 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

Then φ is m.p.t. (HW1) which is ergodic.
(Roy Adler & B. Weiss '73)

Non-examples:

Lemma Nontrivial exchangeable sequences are NOT ergodic

Pf Ex $\{X_k\}$ 0,1-valued, exchangeable, $S_n := X_1 + \dots + X_n$.
de Finetti: $\frac{S_n}{n} \longrightarrow U \in [0,1]$ a.s.
$\Rightarrow U$ is $\mathcal{E}$-measurable $\boxtimes$

Lemma Rational rotations of circle are NOT ergodic.

Pf $\alpha = \frac{p}{q}$ $0 < p \leq q$, $p, q \in \mathbb{N}$, $\gcd(p,q) = 1$

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ \varphi_\alpha^k(x) \longrightarrow \frac{1}{q} \sum_{k=0}^{q} f(x + \tfrac{k}{q} \bmod 1)$$

Note $\mu_a = \frac{1}{q} \sum_{k=0}^{q} \delta_{a + k/q \bmod 1}$

φ_α is ergodic for μ_a: