Markov processes

Goal: Theory of Markov processes with $t \geq 0$ (continuous) and S = general

Setting: $S :=$ compact or locally compact metric (separable) space, $\Sigma := \mathcal{B}(S)$.

$\Omega := \{ \omega \in S^{\mathbb{R}_+} : \text{càdlàg} = \text{RCLL} \}$ $\quad (\omega : [0,\infty) \to S)$,

$X_t(\omega) := \omega(t)$, $\quad \mathcal{F} = \sigma(X_t : t \geq 0)$.

$C(S) := \{ f \in \mathbb{R}^S : \text{continuous} \wedge \forall \varepsilon > 0 \, \exists K \subseteq S \text{ compact} : \sup_{x \notin K} |f(x)| < \varepsilon \}$

$\|f\| := \sup_{x \in S} |f(x)|$, $\quad C(S)$ separable.

Def: A Markov process on S is a family $\{P^x : x \in S\}$ of prob. measures on $(\Omega, \mathcal{F})$ supplied with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ s.t. $\forall t \geq 0 : \mathcal{F}_t \supseteq \sigma(X_s : s \leq t)$, s.t.

(1) $\forall x \in S : P^x(X_0 = x) = 1$

(2) $\forall x \in S \; \forall t, s \geq 0 \; \forall A \in \Sigma$:

$$P^x(X_{t+s} \in A \mid \mathcal{F}_t) = P^{X_t}(X_s \in A) \quad P^x\text{-a.s.} \quad \text{(Markov property)}$$

(3) $\forall f \in C(S) \; \forall t \geq 0 : \quad x \longmapsto E^x(f(X_t)) \in C(S)$. $\quad$ (Feller property)

Thm Let X = M.P. For $t \geq 0$, $f \in C(S)$ define $T(t)f(x) := E^x(f(X_t))$.
Then $T(t)$ is linear operator $C(S) \to C(S)$ and $\forall t,s \geq 0$ $\forall f \in C(S)$ we have:

(1) $T(0)f = f$

(2) $f \geq 0 \Rightarrow T(t)f \geq 0$ (pos. preserving)

(3) $T(t+s)f = T(t)T(s)f$ (semigroup)

(4) $\lim_{t \downarrow 0} \|T(t)f - f\| = 0$ (strong right continuity)

(5) $\|T(t)f\| \leq \|f\|$ (contraction)

(6) $t \mapsto T(t)f$ is (strongly) continuous.

Pf: (1), (2) triv., for (5): $-\|f\| \leq f \leq \|f\| \Rightarrow -\|f\| \leq T(t)f \leq \|f\|$

(3): need Markov property

$$T(t+s)f(x) = E^x(f(X_{t+s})) = E^x(E(f(X_{t+s}) \mid \mathcal{F}_t))$$

$$\overset{MP}{=} E^x(E^{X_t}(f(X_s))) = E^x(T(s)f(X_t)) = T(t)T(s)f(x)$$

(4): RC of paths gives $T(t)f(x) \xrightarrow[t \downarrow 0]{} f(x)$ $\forall x \in S$.

Need:

Lemma $\forall \lambda > 0$ $\forall f \in C(S)$ $\forall x \in S$: $R(\lambda)f(x) := \int_0^\infty e^{-\lambda t} T(t)f(x)\, dt$ defines $R(\lambda)f \in C(S)$

s.t. $\{\lambda R(\lambda)f : f \in C(S), \lambda > 0\}$ is dense in $C(S)$

Pf of Lemma: Existence & continuity checked from properties of integral

$$\lambda R(\lambda) f(x) = \int_0^\infty \lambda e^{-\lambda t} T(t) f(x)\, dt \xrightarrow[\lambda \to \infty]{} f(x)$$

BCT $\forall \mu \in M_1(\hat{S})$: $\int \lambda R(\lambda) f \, d\mu \to \int f \, d\mu$

Riesz $\lambda R(\lambda) f \to f$ weakly

So weak closure of $\{ \dots \}$ is $C(\hat{S})$.

Hahn-Banach weak closure of $\{ \dots \}$ is the (strong) closure of $\{ \dots \}$. $\boxtimes$

Now observe: for $g := \lambda R(\lambda) f$ we have

$$T(t) g = \lambda T(t) R(\lambda) f \overset{\text{Fubini}}{=} \lambda \int_0^\infty e^{-\lambda s} T(t+s) f \, ds$$

$$= \lambda e^{\lambda t} \int_t^\infty e^{-\lambda u} T(u) f \, du = e^{\lambda t} g - e^{\lambda t} \int_0^t e^{-\lambda u} T(u) f \, du$$

$$\| T(t) g - g \| \le (e^{\lambda t} - 1)(\|g\| + \|f\|) \xrightarrow[t \downarrow 0]{} 0$$

3ε-argument proves (4).

For (6): $\| T(t) f - T(s) f \| \overset{(3,5)}{\le} \| T(|t-s|) f - f \|$ $\boxtimes$

Examples of semigroups $S = \mathbb{R}$

(1) $T(t)f(x) := f(x+t)$

(2) $T(t)f(x) := \mathbb{E}\big(f(x+N(0,t))\big) \overset{t>0}{=} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2t}} f(y) \frac{dy}{\sqrt{2\pi t}}$ (Brownian semigroup)

(3) $T(t)f(x) := \int_0^{\infty} \frac{f(x+y)+f(x-y)}{\pi(t^2+y^2)}\, dy$ (Cauchy semigroup)

Thm Suppose $\{T(t) : t \geq 0\}$ are linear operators on $C(S)$ satisfying (1–6) above.
Assume also $\exists \{h_n\}_{n\in\mathbb{Z}} \subseteq C(S)$ $\forall f \in C(S)$ $\forall t \geq 0$: $T(t)(h_n f) \longrightarrow T(t)f$
Then $\exists!$ Markov process s.t.
$\forall t \geq 0$ $\forall x \in S$ $\forall f \in C(S)$: $T(t)f(x) = E^x\big(f(X_t)\big)$.

Lemma $\forall x \in S$ $\exists P^x$ = prob. measure on $(S^{\mathbb{Q}^+}, \Sigma^{\otimes \mathbb{Q}^+})$

s.t. $\tilde{X}_t(\omega) = \omega(t)$ obey:

$\forall n \geq 1$ $\forall t_0, t_1 \ldots t_n \in \mathbb{Q}^+$ s.t. $0 = t_0 < t_1 < \cdots < t_n$ $\forall f_0, \ldots, f_n \in C(S)$,

$$E^x\left(\prod_{i=0}^{n} f_i(X_{t_i})\right) = T(t_0)\Big(f_0\, T(t_1 - t_0)\Big(f_1\, T(t_2 - t_1)\Big(\cdots \Big(T(t_n - t_{n-1}) f_n\Big)\cdots\Big)\Big)\Big)$$

Pf Argument proving Riesz repr. theorem shows $\exists \mu_{t_0 \ldots t_n} \in \mathcal{M}_1(S^{n+1})$

$$\text{RHS} = \int \mu_{t_0 \ldots t_n}(dx_0 \ldots dx_n) \prod_{i=0}^{n} f_i(x_i)$$

These measures are Kolmogorov consistent

KET $\{\mu_{t_1 \ldots t_n}\}$ extend to prob. measure on $(S^{\mathbb{Q}^+}, \Sigma^{\otimes \mathbb{Q}^+})$.

Then identity holds. $\boxtimes$