Wrapping up CTMC's

last time: From X to Q:

- $T := \inf\{t \geq 0 : X_t \neq X_0\}$ ... $\exists\, \bar{c}(x) \in [0,\infty)$: $P^x(T>t) = e^{-\bar{c}(x)t}$
- $\bar{c}(x) > 0 \Rightarrow T' := \inf\{t \geq 0 : X_{T+t} \neq X_T\}$ (*)

  and $P^x(T>t, X_T = y, T'>s) = e^{-\bar{c}(x)t - \bar{c}(y)s} P^x(X_T = y)$

Thm Let $X = $ CTMC. Then no states are instantaneous or defective.
so Q-matrix well defined. Moreover:

$$\forall x \in S: \quad \bar{c}(x) = c(x)$$

$$\forall x,y \in S\ \forall t \geq 0: \quad P^x(T>t, X_T = y) = e^{-c(x)t} \frac{q(x,y)}{c(x)}$$

(if $c(x) > 0$)

Finally, p^* is stochastic.

Pf: $P^x(X_t = x) \geq P^x(T > t) = e^{-\bar{c}(x)t}$

$$\frac{1 - P^x(X_t = x)}{t} \leq \frac{1 - e^{-\bar{c}(t)}}{t} \leq \bar{c}(x)$$

$\Rightarrow \quad c(x) \leq \bar{c}(x)$

Since $\bar{c}(x) < \infty \Rightarrow x$ not instantaneous.

So $g(x,y)$ well-defined and $\sum_{y \neq x} g(x,y) \leq c(x)$

(*) gives: ($y \neq x$, $\bar{c}(x) > 0$)

$$P^x(X_t = y) \geq P^x(T < t, X_T = y, T' > t)$$

$$= (1 - e^{-\bar{c}(x)t}) e^{-\bar{c}(y)t} P^x(X_T = y)$$

So

$$g(x,y) \geq \bar{c}(x) P^x(X_T = y)$$

Then:

$$C(x) \geq \sum_{y \neq x} g(x,y)$$

$$\geq \bar{C}(x) \sum_{y \neq x} P^x(X_T = y) = \bar{C}(x) \geq C(x)$$

Hence, all have to be equalities, i.e.

$$\forall y \neq x: \quad g(x,y) = \bar{C}(x) P^x(X_T = y)$$

$$\forall x \in S: \quad \bar{C}(x) = C(x)$$

Summary:

$X \to P, Q$ no caveats

$P \to Q$ instantaneous, defective

$Q \to P, X$ explosion

$P \to X$ all above (Blackwell)

Thm (Forward Kolmogorov equation)

Let $p^* =$ min solution to BKE. Then

$\forall x, y \in S \ \forall t \geq 0$: $\displaystyle\sum_{z \neq x} g(x,z) p^*_t(z,y) = \sum_{z \neq y} p^*_t(x,z) g(z,y)$

In particular,

$$\frac{d}{dt} p^*_t(x,y) = \sum_{z \in S} p^*_t(x,z) g(z,y)$$

holds, $\forall t \geq 0 \ \forall x, y \in S$.

FKE

Lemma $\forall t \geq 0\ \forall x, y \in S\ \forall n \geq 0$:

$$p_t^{(n+1)}(x,y) = e^{-c(x)t}\delta_{xy} + \int_0^t e^{-c(x)(t-s)} \sum_{z \neq y} p_t^{(n)}(x,z)\, g(z,y)\, ds$$

Pf Unwrap def of $p^{(n)}$:

$$p_t^{(n+1)}(x,y) = e^{-c(x)t}\delta_{xy} + \sum_{k=1}^{n} \sum_{\substack{x_0,\dots,x_{n+1} \\ x_0 = x,\ x_{n+1} = y}} \int_{0 = t_0 < \dots < t_n < t_{n+1} = t} e^{-\sum_{i=1}^{n+1} c(x_{i-1})(t_i - t_{i-1})} \prod_{i=1}^{n+1} g(x_{i-1}, x_i)\, dt_1 \dots dt_n$$

$\forall i = 1 \dots n+1:\ x_i \neq x_{i-1}$

Now keep x_n fixed and rearr the rest.

Corollary

$$\forall t \geq 0\ \forall x, y \in S: \quad p_t^*(x,y) = e^{-c(x)t}\delta_{xy} + \int_0^t e^{-c(x)(t-s)} \sum_{z \neq x} p_t^*(x,z)\, g(z,y)\, ds$$

Pf of Thm: By Lebesgue differentiation...

$(**)(t)$ holds for Leb.-a.e. $t \geq 0$.

Given $u \geq t$, note: $p^*_u(x,z) \geq p^*_{u-t}(x,x)\, p^*_t(x,z)$ by Ch-K.

So

$$\sum_{z \neq y} p^*_u(x,z)\, \underline{g}(z,y) \geq p^*_{u-t}(x,x) \sum_{z \neq y} p^*_t(x,z)\, \underline{g}(z,y)$$

Taking $u \downarrow t$ or $t \uparrow u$ then proves $(**)(t)$ for all $t > 0$. At $t = 0$, equality holds by fiat.

$\boxtimes$

...:

...0.

$...x,x)\, p^*_t(x,z)$

...k.

$...y)$

$...(t)$

...by fiat.

$\boxed{\times}$

Semigroup formulation of BKE/FKE:

$f \in \ell^\infty(S)$:

$$\sum_{z\in S} q(x,z)\, f(z) = \sum_{z\neq x} q(x,z)\, f(z) - c(x)\, f(x)$$

$$= \sum_{z\in S} q(x,z)\,[f(z) - f(x)]$$

$$=: Lf(x)$$

linear operator on $\ell^\infty(S)$.

<u>generator</u>

$$T(t)f(x) := E^x(f(X_t))$$

$$= \sum_{y\in S} p_t(x,y)\, f(y)$$

$\{T(t) : t \geq 0\}$ are bdd, linear on $\ell^\infty(S)$.
with <u>semigroup property</u>

$$T(t)\,T(s)f = T(t+s)f \quad \forall s, t \geq 0$$

by Chapman-Kolmogorov.

<u>BKE</u> : $\dfrac{d}{dt} T(t)f = L\,T(t)f$

<u>FKE</u> $\dfrac{d}{dt} T(t)f = T(t)Lf \quad (f \in \mathrm{Dom}(L))$

Def μ is stationary if

$$\forall t \geq 0 \ \forall x \in S: \quad \mu(x) = \sum_{y \in S} \mu(y)\, p_t(y,x)$$

Thm If μ is stationary then

$$\forall x \in S: \quad \sum_{z \in S} \mu(y)\, q(y,x) = 0$$

If also $\sum_{x \in S} \mu(x)\, c(x) < \infty$ then the converse holds as well.