Continuous-time Markov chains

Last time: From Q to P ... p^* = minimal ≥ 0 solution to BKE

$$\sum_{y\in S} p^*_t(x,y) = 1 \iff P^x(N(t)<\infty) = 1 \quad \forall t \geq 0.$$

Def q is non-explosive if this happens $\forall x \in S$

Ex
$$0 \xrightarrow[c(0)]{} 1 \xrightarrow[c(1)]{} 2 \xrightarrow[c(2)]{} 3 \longrightarrow \cdots$$

$q(k,k+1) = c(k)$, $\sum_{k=0}^{\infty} \frac{1}{c(k)} < \infty$

$\Rightarrow$ chain explosive.

$$\boxed{\textbf{Thm } \forall x \in S: \quad \left(\forall t \geq 0 : P^x(N(t)<\infty)=1\right) \iff \sum_{k=0}^{\infty} \frac{1}{c(Z_k)} = +\infty \quad P^x\text{-a.s.}}$$

Pf: $\forall t \geq 0: P^x(N(t) < \infty) = 1$

$$\Leftrightarrow P^x\left(\sum_{k=0}^{\infty} T_k / c(Z_k) = +\infty\right) = 1$$

$$\Leftrightarrow E\left(\exp\left\{-\lambda \sum_{k=0}^{n} T_k / c(Z_k)\right\}\right) \xrightarrow[n\to\infty]{\forall \lambda > 0} 0$$

$$= E^x\left(\prod_{k=0}^{n} \frac{1}{1 + \lambda / c(Z_k)}\right) \longrightarrow \begin{cases} 0 & \text{if } \sum_{k=0}^{\infty} \frac{1}{c(Z_k)} = +\infty \text{ a.s.} \\ > 0 & \text{else} \end{cases}$$

Corollary Q is non-explosive if

(1) $\sup_{x \in S} c(x) < \infty$ or (2) Z is recurrent (and irreducible)

Note Our construction gave us $X_t := Z_{N(t)}$

Markov property

Def Let $\Omega := \{ \omega \in S^{\mathbb{R}_+} : \text{RCLL (càdlàg) \& isolated jumps} \}$

$$X_t(\omega) := \omega(t), \quad \mathcal{F} = \sigma(X_t : t \geq 0).$$

$$\theta_t(\omega)(s) := \omega(t+s) \quad \text{(shift by } t\text{)}$$

A continuous-time Markov chain (CTMC) is a family $\{P^x\}_{x \in S}$ of prob. measures on $(\Omega, \mathcal{F})$, with s.t. for a filtration $\{\mathcal{F}_t\}_{t \geq 0}$

$$\forall t \geq 0: \quad \mathcal{F}_t \supseteq \sigma(X_s : s \leq t)$$

we have:

$$\forall x \in S: \quad P^x(X_0 = x) = 1$$

$$\forall x \in S \; \forall A \in \mathcal{F} \; \forall t \geq 0:$$

Markov property

$$P^x(\underbrace{\theta_t^{-1}(A)}_{X_{t+\cdot} \in A} \mid \mathcal{F}_t) = P^{X_t}(A) \quad P^x\text{-a.s.}$$

Lemma $X_t = Z_{N(t)}$ defined last time is (under non-explosion) a CTMC.

Note For Markov property, suffices to check:

$\forall x \in S\ \forall t, s \geq 0\ \forall y \in S$:

$$P^x(X_{t+s} = y \mid \mathcal{F}_t) = P^{X_t}(X_s = y) \quad P^x\text{-a.s.}$$

So we have a direct passage $Q \longrightarrow X$ (under non-explosion).

Note We also have $X \to P_t$

Lemma If $X = \text{CTMC}$, then $p_t(x,y) = P^x(X_t = y)$ is transition function.

is

$\mathcal{F}$).

a.s.

Now <u>Direct route $X \to Q$</u>

<u>Need</u> Strong Markov property

<u>Def</u> An $\mathbb{R}_+ \cup \{+\infty\}$-valued RV T is <u>stopping time</u> for $\{\mathcal{F}_t\}_{t \geq 0}$ if $\forall t \geq 0$: $\{T \leq t\} \in \mathcal{F}_t$.

<u>Def</u> $\mathcal{F}_T := \{A \in \mathcal{F} : (\forall t \geq 0 : A \cap \{T \leq t\} \in \mathcal{F}_t)\}$

(σ-alg. of events measurable by stopping time T)

<u>Thm</u> (SMP) Let X = CTMC. Then

$\forall x \in S\ \forall t \geq 0\ \forall A \in \mathcal{F}$:

$$P^x\left(\{T<\infty\} \cap \theta_T^{-1}(A) \,\middle|\, \mathcal{F}_T\right) = P^{X_T}(A) \quad P^x\text{-a.s. on } \{T<\infty\}.$$

From X to Q:

Lemma Let $X = $ CTMC. Set $T := \inf\{t \geq 0 : X_t \neq X_0\}$

Then

$\forall x \in S\ \exists \bar{c}(x) \in [0,\infty)\ \forall t \geq 0:\ P^x(T > t) = e^{-\bar{c}(x)t}$

I.e., $T = \text{Exp}(c(x))$

Pf: $P^x(T > t+s) = P^x(T > t \cap \theta_t^{-1}(T > s))$

$\overset{MP}{=} E^x(1_{T>t} P^{X_t}(T > s))$

$= P^x(T > t) P^x(T > s)$

Fekete $\Rightarrow P^x(T > t) = e^{-\bar{c}(x)t}$

Since $P^x(T > t) \xrightarrow[t \downarrow 0]{} 1 \Rightarrow \bar{c}(x) < \infty$. $\boxtimes$

Lemma Let $x \in S$ be s.t. $\bar{c}(x) > 0$.

Define $T' := \inf\{t \geq 0 : X_{T+t} \neq X_T\}$ $(T < \infty)$.

Then

$\forall y \in S \ \forall t, s \geq 0$:

$$P^x(T > t, X_T = y, T' > s)$$

$$= P^x(T > t) P^y(T > s) P^x(X_T = y)$$

Pf: $LHS = E^x(1_{T>t} 1_{X_T = y} \theta_T^{-1}(T > s))$

$\overset{SMP}{=} E^x(1_{T>t} 1_{X_T = y} \cdot P^{X_T}(T > s))$

$= P^y(T > s) P^x(T > t, X_T = y)$

Now

$$P^x(X_T = y, T > t+s) = P^x(T > t \cap \theta_t^{-1}(X_T = y, T > s))$$

$$\overset{MP}{=} E^x(1_{T>t} P^{X_t}(X_T = y, T > s))$$

$$= P^x(X_T = y, T > s)\, P^x(T > t)$$

Now take $s \downarrow 0$ to get

$$P^x(X_T = y, T > t) = P^x(X_T = y)\, P^x(T > t)$$

$\boxtimes$

Thm: Suppose $X = CTMC$ (which defines P_t)

Then no states are instantaneous or defective (so get Q)

Moreover, $\forall x \in S: \; c(x) = \bar{c}(x)$

and $\forall x, y \in S \; \forall t \geq 0: \; P^x(T > t, X_T = y) = \dfrac{q(x,y)}{c(x)} e^{-c(x)t}$

(So X is the chain we get for Q). Moreover, q is non-explosive