Continuing the continuous-time Markov chain (CTMC)

transition function $\left\{ \begin{array}{l} p_t(\cdot,\cdot) \geq 0, \ \sum_{y \in S} p_t(x,y) = 1, \ \sum_z p_t(x,z) p_s(z,y) = p_{t+s}(x,y) \\ p_t(x,y) \xrightarrow[t \downarrow 0]{} \delta_{xy}. \end{array} \right.$

Q-matrix $\longrightarrow$ $q(x,y) \geq 0 \quad x \neq y$

$\sum_{y \in S} q(x,y) = 0.$

idea: $p_t(x,y) \overset{\text{WANT}}{=} P^x(X_t = y)$

last time: $c(x) = -q(x,x) := \lim_{t \downarrow 0} \frac{1 - p_t(x,x)}{t}$ exists in $[0,\infty]$

$c(x) < \infty \Rightarrow q(x,y) \underset{y \neq x}{=} \frac{d}{dt} p_t(x,y)\Big|_{t=0}$ — " — & $\sum_{y \in S} q(x,y) \leq 0.$

(<0 possible if $|S| = \infty$)

if $c(x) < \infty \wedge \sum_{y \in S} q(x,y) = 0 \Rightarrow$ Backward Kolmogorov eq:

$$\frac{d}{dt} p_t(x,y) = \sum_{z \in S} q(x,z) p_t(z,y) \quad (t \geq 0)$$

Note BKE integral form:

$$p_t(x,y) = e^{-c(x)t} \delta_{xy} + \int_0^t e^{-c(x)(t-s)} \underbrace{\sum_{z \neq x} q(x,z) p_s(z,y)}_{\geq 0}$$

Lemma Suppose $x \in S$ is s.t. $\limsup_{t \downarrow 0} \sup_{y \neq x} P_t\cancel{(x,y)}(y,x) < 1$.

Then $c(x) < \infty$.

Pf: HW.

Corollary Suppose S is finite. Then there are no instantaneous or defective states for any transition function on S. Moreover, transition functions and Q-matrices are in 1-1 correspondence via

$$P_t = e^{tQ} := \sum_{n \geq 0} \frac{1}{n!} t^n Q^n$$

Pf: Lemma + continuity $\Rightarrow$ $\forall x \in S$: $c(x) < \infty$.
Fatou defect does NOT occur.
Then P_t is tr. function (need argument in next proof).

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Thm Suppose a Q-matrix obeys $\sup_{x \in S} (-q(x,x)) < \infty$.
Then q arises from transition function.

Pf: Set $\lambda := \left[\sup_{x \in S} (-q(x,x)) \right]^{-1}$.

Then $\overline{P} := I + \lambda Q$. Then P is stochastic, i.e. transition kernel of MC.

Let $N(t) :=$ homogeneous Poisson process (rate 1)

Let $Z = \{Z_n\}_{n \geq 0}$ = M.C. with tr. kernel $\overline{P}$. Define $X_t := Z_{N(t/\lambda)}$

Define $p_t(x,y) = P^x(X_t = y)$.

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P is ... of MC.

... process ...-1) ...th

$\sum N(t/\lambda)$ ...).

Then $p_t(x,\cdot)$ = prob. mass. function,

$p_t(x,y) \xrightarrow[t \downarrow 0]{} \delta_{xy}$ because $N(t/\lambda) \xrightarrow[t \downarrow 0]{} 0$.

Chapman-Kolmogorov:

$$\begin{aligned} p_{t+s}(x,y) &= P^x(X_{t+s}=y) \\ &= \sum_z P^x(X_t = z, X_{t+s}=y) \\ &= \sum_z \sum_{n,m\geq 0} \frac{(t/\lambda)^n}{n!} e^{-t/\lambda} \frac{(s/\lambda)^m}{m!} e^{-s/\lambda} \bar{P}^n(x,z) \bar{P}^m(z,y) \\ &= \sum_z P^x(X_t = z) P^z(X_s = y) = \sum_z p_t(x,z) p_s(z,y) \end{aligned}$$

Finally:

$$\begin{aligned} \frac{d}{dt} p_t(x,y) &= \frac{d}{dt} \sum_{n\geq 0} \frac{(t/\lambda)^n}{n!} e^{-t/\lambda} \bar{P}^n(x,y) \\ &= -\frac{1}{\lambda} p_t(x,y) + \frac{1}{\lambda} \sum_{z\in S} \bar{P}(x,z) p_t(z,y) \\ &\xrightarrow[t\downarrow 0]{} -\frac{1}{\lambda}\delta_{xy} + \frac{1}{\lambda}\bar{P}(x,y) = \frac{1}{\lambda}(\bar{P}-1)(x,y) = Q(x,y) \end{aligned}$$

$\boxtimes$

GOAL Develop a theory with unbounded jump rates.

IDEA Take q matrix & construct a solution to BKE.

And only then worry about its properties.

Def Let

$$p_t^{(0)}(x,y) := 0$$

$$p_t^{(n+1)}(x,y) := e^{-c(x)t}\delta_{xy} + \int_0^t e^{-c(x)(t-s)} \sum_{z \neq x} q(x,z)\, p_s^{(n)}(z,y)$$

Lemma

- $\forall t \geq 0\ \forall x,y \in S\ \forall n \geq 0$: $p_t^{(n+1)}(x,y) \geq p_t^{(n)}(x,y)$
- $\forall t \geq 0\ \forall x \in S\ \forall n \geq 0$: $\sum_{y \in S} p_t^{(n)}(x,y) \leq 1$
- Setting $p_t^*(x,y) := \lim_{n \to \infty} p_t^{(n)}(x,y)$, we have

$$p_t^*(x,y) = e^{-c(x)t}\delta_{xy} + \int_0^t e^{-c(x)(t-s)} \sum_{z \neq x} q(x,z)\, p_t^*(z,y)$$

<u>Lemma</u> Every non-negative solution $p_t(x,y)$ of BKE obeys

$\forall t \geq 0 \ \forall x, y \in S: \quad p_t(x,y) \geq p_t^*(x,y).$

Hence, if $\sum_{y \in S} p_t^*(x,y) = 1$, then p^* is the <u>unique</u> transition function obeying BKE.

<u>Pf</u> 1^{st} part by induction from $p_t(x,y) \geq 0 = p_t^{(0)}(x,y)$.

2^{nd} part: Sum inequality to get

$$1 = \sum_{y \in S} p_t(x,y) \geq \sum_{y \in S} p_t^*(x,y) \overset{\text{assume}}{=} 1$$

So $=$ held above. $\boxtimes$

Lemma Define
$$\bar{P}(x,y) := \begin{cases} \frac{q(x,y)}{c(x)} 1_{x \neq y} & \text{if } c(x) > 0 \\ \delta_{xy} & \text{if } c(x) = 0 \end{cases}$$

Let P^x be joint law of discrete-time MC Z and iid $\mathrm{Exp}(1)$ $\{T_k\}_{k \geq 0}$. Given a path of Z, let
$$N(t) := \sup\left\{ m \geq 0 : \sum_{k=1}^{m} T_k / c(Z_k) \leq t \right\}.$$
and let $X_t := Z_{N(t)}$ on $\{N(t) < \infty\}$. Then:

$\forall t \geq 0$ $\forall x, y \in S$:
$$\forall n \geq 0 : \quad p_t^{(n)}(x,y) = P^x(X_t = y, N(t) < n)$$
and so
$$p_t^*(x,y) = P^x(X_t = y, N(t) < \infty)$$
In particular,
$$\sum_{y \in S} p_t^*(x,y) = \boxed{P^x(N(t) < \infty)}.$$

Pf: $p_t^{(0)}(x,y) = 0 = P^x(X_t = y, N(t) < 0)$

$$P^x(X_t = y, N(t) < n+1)$$

$$= P^x(X_t = y, T_0/c(x) \geq t)$$

$$+ P^x(X_t = y, T_0/c(x) < t, N(t) < n+1)$$

$$= \delta_{xy} e^{-c(x)t}$$

$$+ \int_0^t ds\, c(x)\, e^{-c(x)(t-s)} \sum_z \underbrace{\bar{P}(x,z)}_{= \frac{q(x,z)}{c(x)}} P^z(X_s = y, N(t) < n)$$

Plug in identity & iterate. $\boxtimes$

<u>Lemma</u> p^x obeys Chapman-Kolmogorov.