Continuous time M.C's

transition function p_t PMF + Chapman-Kolmogorov + continuity at $t=0$

Q-matrix $\forall x,y\in S: (x\neq y \Rightarrow q(x,y)\geq 0) \wedge \sum_{z\in S} q(x,z)=0$

Ex $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$ $\alpha,\beta\geq 0$ ... $Q\begin{pmatrix}1\\1\end{pmatrix}=0$, $Q\begin{pmatrix}-\alpha\\ \beta\end{pmatrix} = -(\alpha+\beta)\begin{pmatrix}-\alpha\\ \beta\end{pmatrix}$

$p_t(x,0)+p_t(x,1)=1$, $-\alpha p_t(x,0)+\beta p_t(x,1)$
$= e^{-(\alpha+\beta)t}(-\alpha\delta_{x,0}+\beta\delta_{x,1})$

$$P_t = e^{tQ}$$

$$= \begin{pmatrix} \frac{\beta}{\alpha+\beta}+\frac{\alpha}{\alpha+\beta}e^{-(\alpha+\beta)t} & , & \frac{\alpha}{\alpha+\beta}\left(1-e^{-t(\alpha+\beta)}\right) \\ \frac{\beta}{\alpha+\beta}\left(1-e^{-t(\alpha+\beta)}\right) & , & \frac{\alpha}{\alpha+\beta}+\frac{\beta}{\alpha+\beta}e^{-t(\alpha+\beta)} \end{pmatrix}$$

<u>Today</u>: From p_t to Q:

<u>Thm</u> Suppose p_t is transition function. Then

(1) $\forall x \in S$:

$$c(x) := -q(x,x) = \frac{d}{dt} p_t(x,x) \Big|_{t=0^+} \text{ exists in } [0,\infty]$$

and we have $p_t(x,x) \geq e^{-t c(x)}$,

(2) If $c(x) < \infty$, then

$$\forall y \in S: \ q(x,y) := \frac{d}{dt} p_t(x,y) \Big|_{t=0^+} \text{ exists and}$$

obeys $\sum_{z \in S} q(x,z) \leq 0$.

(3) if $c(x) < \infty \wedge \sum_{z \in S} q(x,z) = 0$ then $t \mapsto p_t(x,y) \in C^1$ $\forall y \in S$

Backward Kolmogorov equation

and $$\frac{d}{dt} p_t(x,y) = \sum_{z \in S} q(x,z) p_t(z,y) \quad \begin{matrix} y \in S \\ t \geq 0. \end{matrix}$$

De

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n

s in $[0,\infty]$

and

$\forall y \in S$

$y) \in C'$

$y \in S$

$t \geq 0$.

Def $x \in S$ is

- instantaneous if $c(x) = +\infty$
- stable if $c(x) < \infty$.
- absorbing if $c(x) = 0$
- defective if $c(x) < \infty \wedge \sum_{y \in S} q(x,y) < 0$

Lemma (1) $\forall x \in S\ \forall t,s \geq 0 : p_{t+s}(x,x) \geq p_t(x,x)\, p_s(x,x)$

Hence: (2) • $\exists t > 0\ p_t(x,x) = 1 \Rightarrow \forall t \geq 0\ p_t(x,x) = 1$

(3) • $\forall t \geq 0\ p_t(x,x) > 0$

Pf (1) by Chapman-Kolmogorov

(2) Need to show that p_t(x,x) = 0 implies p_{t/2}(x,x)=1
Then argue by continuity proved in the next lemma

(3) TRUE for t small by continuity + iterate.

Lemma

Pf

Pf

Lem

$\underline{\text{Pf}}$ Denote $c := \sup_{t>0} \frac{f(t)}{t}$. $\underline{\text{NTS}}$ $\liminf_{t\downarrow 0} \frac{f(t)}{t} \geq c$.

Assume $c < \infty$ and given $\varepsilon > 0$ pick t_0: $\frac{f(t_0)}{t_0} \geq c - \varepsilon$.

Then
$$f(t_0) \underset{t<t_0}{\leq} \left\lfloor \frac{t_0}{t} \right\rfloor f(t) + \underbrace{f\left(t_0 - \left\lfloor \frac{t_0}{t} \right\rfloor t\right)}_{\leq ct}$$

So
$$\frac{f(t)}{t} \geq \frac{t_0}{\left\lfloor \frac{t_0}{t} \right\rfloor t} \underbrace{\frac{f(t_0)}{t_0}}_{\geq c-\varepsilon} - \frac{c}{\left\lfloor \frac{t_0}{t} \right\rfloor}$$

$$\underset{t\downarrow 0}{\geq} c - \varepsilon \xrightarrow[\varepsilon \downarrow 0]{} c$$

For $c = +\infty$, use instead $\frac{f(t_0)}{t_0} \geq M \Rightarrow \liminf_{t\downarrow 0} \frac{f(t)}{t} \geq M$. $\boxtimes$

Proof of (2) Assume $c(x)<\infty$. Then $1-p_t(x,x) \le 1-e^{-c(x)t} \le c(x)\,t$

So $p_t(x,y) \le c(x)\,t$ $\forall y\ne x$. So $g(x,y) := \limsup_{t\downarrow 0} \frac{p_t(x,y)}{t} \le c(x)$, $y \ne x$.

NTS liminf is the same. By Chapman-Kolmogorov:

Pick $t>\delta>0$, $n = \lfloor t/\delta \rfloor$

$$p_t(x,y) = \sum_{\substack{z_1\dots z_n\in S \\ z_0=x}} \left(\prod_{k=1}^{n} p_\delta(z_{k-1},z_k)\right) p_{t-n\delta}(z_n,y)$$

$$\underset{y\ne x}{\ge} \sum_{k=0}^{n-1} p_\delta(x,x)^k\, p_\delta(x,y)\, p_{t-(k+1)\delta}(y,y)$$

$$p_t(x,y) \ge n\, e^{-c(x)t}\, p_\delta(x,y) \inf_{s\le t} p_s(y,y)$$

$$\frac{p_t(x,y)}{t} \ge e^{-c(x)t}\inf_{s\le t} p_s(y,y)\, \frac{n\delta}{t}\,\frac{p_\delta(x,y)}{\delta} \xrightarrow[\substack{\delta\downarrow 0 \\ \text{(subsequence)}}]{} e^{-c(x)t}\underbrace{\inf_{s\le t}p_s(y,y)}_{\xrightarrow[t\downarrow0]{}1}\, g(x,y)$$

to get $\liminf_{t\downarrow0}\frac{p_t(x,y)}{t}\ge g(x,y)$. Fatou's lemma gives $\sum_{y\in S} g(x,y)\le 0$

$y)$

$\sum_{z\notin T} g(x,z)$

$\boxtimes$

$g(x,z)$

Ex A M.C. with all states instantaneous

due to Blackwell.

Take ∞-sequence of 2-state M.C's. with "rates" (α_k,β_k).

$$\bar{P}(\underline{x},\underline{y}) = \prod_{k\geq 1} P^{(\alpha_k,\beta_k)}(x_k,y_k)$$

matrix from earlier

$\underline{x},\underline{y}\in\{0,1\}^{\mathbb{N}}$.

lem Let $S := \{\underline{x}\in\{0,1\}^{\mathbb{N}} : \sum_{i\geq 1} x_i < \infty\}$

Then

$$\sum_{k\geq 1}\frac{\alpha_k}{\alpha_k+\beta_k} < \infty \Rightarrow \forall \underline{x}\in S\ \forall t\geq 0 : \sum_{\underline{y}\in S} \bar{P}_t(\underline{x},\underline{y}) = 1$$

In fact, $\bar{P}$ is tr. function on S

Pf $p^{(\alpha,\beta)}(0,1) = \frac{\alpha}{\alpha+\beta}\left(1 - e^{-(\alpha+\beta)t}\right) \le \frac{\alpha}{\alpha+\beta}$.

$$\sum_{\underline{y} \in S} \overline{p}(\underline{x}, \underline{y}) \underset{\substack{x_k = 0 \\ k \ge n}}{\ge} \prod_{k \ge n} p^{(\alpha_k, \beta_k)}(0,0)$$

$$\ge \prod_{k \ge n} \left(1 - \frac{\alpha_k}{\alpha_k + \beta_k}\right) \xrightarrow[n \to \infty]{} 1.$$

Claim $\sum_{k \ge 1} \alpha_k = +\infty \Rightarrow \forall x \in S : c(x) = +\infty$.

Pf $1 - \overline{p}_t(x,x) \underset{\substack{x_k = 0 \\ k \ge m}}{\ge} 1 - \prod_{m \le k \le n} \underbrace{p_t^{(\alpha_k, \beta_k)}(0,0)}_{\left(1 - \alpha_k t + o(t)\right)}$

$$\frac{1 - \overline{p}_t(x,x)}{t} \ge \left(\sum_{k=m}^{n} \alpha_k\right) + o(1)$$

$$\Rightarrow c(x) \ge \sum_{k=m}^{n} \alpha_k \xrightarrow[n \to \infty]{} \infty. \qquad \boxtimes$$