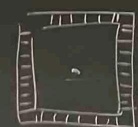
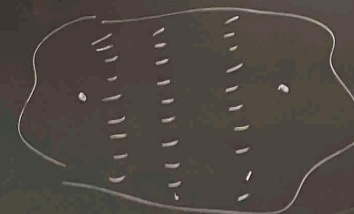
Wrapping-up electrostatic connections

recall Nash-Williams: if $\{E_k : k \geq 1\}$ cutsets of u from v, disjoint, then

$$R_{eff}(u,v) \geq \sum_{k\geq 1} \frac{1}{\sum_{(x,y)\in E} c(x,y)}$$





Cor SRW on $\mathbb{Z}^d$, $d=1,2$, is recurrent.

Pf $\Lambda_n = [-n,n]^d \cap \mathbb{Z}^d$, $E_n = \{(x,y)\in E(\mathbb{Z}^d) : x\in\Lambda_n, y\notin\Lambda_n\}$

$$\sum_{(x,y)\in E_n} c(x,y) = |E_n| \leq c\, n^{d-1} \quad \dots \quad R_{eff}(0,\infty) \geq \sum_{n\geq 1} \frac{1}{c\, n^{d-1}} = +\infty.$$

series laws if nested

For transience, NTS $R(u,\infty) < \infty$. Enough: produce a unit flow to ∞ of finite energy

Lemma (Terry Lyons' random-path method) $P = (x_0, x_1, \dots, x_{|P|})$

Let $\mathbb{P}$ = prob measure on paths in $(V, E(V))$ from u to v.

Then $$i(x,y) := \mathbb{E}\left(\sum_{k=1}^{|P|} \left[\mathbb{1}_{(x,y)=(x_{k-1},x_k)} - \mathbb{1}_{(x,y)=(x_k,x_{k-1})} \right] \right)$$

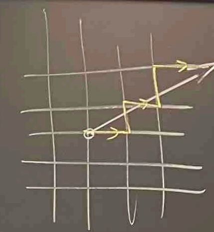
defines $i \in \Phi_{u,v}$ with $|val(i)| = 1$. (Assuming exp. exists). In particular, $R_{eff}(u,v) \leq \mathcal{E}(i)$

Moreover, if $P = (X_0, \dots, X_{\tau_v})$ where X = sample from $\mathbb{P}^u$, then $R_{eff}(u,v) = \mathcal{E}(i)$.

Cor SRW on $\mathbb{Z}^d$, $d \geq 3$, is transient

Pick t uniform for unit sphere in $\mathbb{R}^d$.

Let $P = (x_0 = 0, x_1, \dots)$ be s.t. $\sum_{k \geq 0} 4^{-k} \|x_k - t \cdot x_k\|$ is minimal



Let i be as above. Then

$$\exists c > 0 \ \forall (x,y) \in E(\mathbb{Z}^d) \cap (\Lambda_n \times \Lambda_n^c) : |i(x,y)| \leq \frac{c}{n^{d-1}}$$

Hence $$\mathcal{E}(i) = \sum_{n \geq 1} \sum_{\substack{(x,y): \\ x \in \Lambda_n, y \notin \Lambda_n}} i(x,y)^2 \leq \sum_{n \geq 1} \hat{c}\, n^{d-1} \left(\frac{c}{n^{d-1}}\right)^2 = \hat{c} \sum_{n \geq 1} \frac{1}{n^{d-1}} < \infty \quad d \geq 3.$$

$$\mathbb{E} R_{eff}(u,v) \leq \mathbb{E}\left(\sum_{(x,y)} r(x,y)\, i(x,y)^2\right) \leq \sum_{(x,y)} \left(\mathbb{E}\, r(x,y)\right) i(x,y)^2$$

Thm (Rayleigh monotonicity principle) Let $c, \tilde{c} : E(V) \to \mathbb{R}_+$ be conductances s.t.

$$\forall (x,y) \in E(V) : c(x,y) \leq \tilde{c}(x,y)$$

Let X and $\tilde{X}$ be associated M.C's. Then

$$X \text{ transient} \Rightarrow \tilde{X} \text{ transient}.$$

Pf $r(x,y) \geq \tilde{r}(x,y) \Rightarrow R_{eff}(u,v) \geq \tilde{R}(u,v)$

Ex If a graph embeds a transient graph, it is transient.

Ex Suppose conductance chosen @ random. On $\mathbb{Z}^d$

If $\sup_{(x,y) \in E(V)} \mathbb{E}\, c(x,y) < \infty \Rightarrow X$ recurrent in $d = 1,2$

$\sup_{(x,y) \in E(V)} \mathbb{E} \frac{1}{c(x,y)} < \infty \Rightarrow X$ transient in $d \geq 3$.

Continuous-time Markov chains

Q: How do we make a M.C. run in continuous time?

S: Take discrete-time M.C. & move according to renewal process

i.e., $X_t = Z_{N(t)}$

Issue: this will NOT be Markov (sense below) unless $N(t)$ is Poisson process.

Lemma Let Z = discrete time M.C. on countable S, $\{N(t): t \geq 0\}$ = homogeneous Poisson process independent of Z. Set $X_t := Z_{N(t)}$ and $\mathcal{F}_t = \sigma(X_s : s \leq t)$.

Then $\forall x \in S$ $\forall t, u \geq 0$

$$P^x(X_{t+u} \in \cdot \mid \mathcal{F}_t) = P^{X_t}(X_u \in \cdot) \quad P^x\text{-a.s.}$$

Pf Let $A \in \mathcal{F}_t$. Then $\forall y \in S$:

$$E^x\left(1_{X_{t+u}=y}\, 1_A\right) = \sum_{n,m\ge 0}\sum_{z\in S} E^x\Big(\underbrace{1_{X_{t+u}=y}}_{Z_{n+m}=y}\, \underbrace{1_{N(t+u)-N(t)=m}}_{\perp\!\!\!\perp \{N(s):\, s\le t\}}\, 1_{N(t)=n}\, \underbrace{1_{X_t=z}}_{Z_n=z}\, 1_A\Big)$$

$A\cap\{N_t=n\} \in \sigma\big(N(s): s\le t,\ Z_0,\dots,Z_n\big)$

$$= \sum_{n,m\ge 0}\sum_{z\in S} P^m(z,y)\, \underbrace{P\big(N(t+u)-N(t)=m\big)}_{P(N(u)=m)}\, E^x\left(1_{N(t)=n}\, 1_{X_t=z}\, 1_A\right)$$

$$= \sum_{n\ge 0}\sum_{z\in S} P^z(X_u=y)\, E^x\left(1_{N(t)=n}\, 1_{X_t=z}\, 1_A\right)$$

$$= E^x\left(P^{X_t}(X_u=y)\, 1_A\right). \quad (X_t \text{ is } \mathcal{F}_t\text{-meas.})$$

Q: Is this the only way? A: Well, we can make the "clock" depend on Z!

Def A transition function is a collection $\{p_t(x,y):\ t\ge 0,\ x,y\in S\}$ s.t.

(1) $\forall t\ge 0\ \forall x,y\in S$: $p_t(x,y)\ge 0 \wedge \sum_{z\in S} p_t(x,z)=1$

(2) $\forall t,s\ge 0\ \forall x,y\in S$: $p_{t+s}(x,y) = \sum_{z\in S} p_t(x,z)\, p_s(z,y)$

Chapman-Kolmogorov equations

(3) $\forall x\in S$: $\lim_{t\downarrow 0} p_t(x,x) = p_0(x,x) = 1$