Reversible M.C's & electrostatics

$(V, E(V))$ = undirected conn. graph, $c : E(V) \to \mathbb{R}_+$, $r : E(V) \to \mathbb{R}_+$, $r(x,y) = \dfrac{1}{c(x,y)}$

$$\mathcal{E}(f) := \frac{1}{2} \sum_{(x,y)\in E(V)} c(x,y)\,[f(y)-f(x)]^2 \rightsquigarrow C_{eff}(u,v) := \inf\{\mathcal{E}(f) : f(u)=1, f(v)=0\}$$

minimized by $f_*(x) = P^x(\tau_u < \tau_v)$ $\quad x \neq u,v$

Lemma $\quad C_{eff}(u,v) = \pi(u)\, P^u(\tau_v < \hat{\tau}_u)$ where $\hat{\tau}_x := \inf\{\underline{n \geq 1} : X_n = x\}$

Pf $\quad \mathcal{E}(f_*) = (f_*, (I-P)f_*)_{\ell^2(\pi)} \underset{\substack{Pf = f \\ \text{on } V\setminus\{u,v\}}}{=} \pi(u)\underset{=1}{f_*(u)}\,(I-P)f_*(u) + \pi(v)\underset{=0}{f_*(v)}\,(I-P)f_*(v)$

$$(I-P)f_*(u) = f_*(u) - Pf_*(u)$$

$$= 1 - \sum_{x\in V} P(u,x)\,P^x(\tau_x < \tau_v) = 1 - P^u(\hat{\tau}_u < \tau_v) = P^u(\tau_v < \hat{\tau}_u).$$

$$\mathcal{I}_{u,v} = \left\{ i : E(V) \to \mathbb{R} : \begin{array}{l} \forall (x,y) \in E(V): \ i(x,y) = -i(y,x) \\ \forall x \neq u,v: \ \sum_{y:(x,y)\in E(V)} i(x,y) = 0 \end{array} \right\}$$

"flow from u to v".

$$\mathrm{val}(i) := \sum_{y:\,(u,y)\in E(V)} i(u,y) \qquad \text{value of flow}$$

$$\tilde{\mathcal{E}}(i) = \frac{1}{2} \sum_{(x,y)\in E(V)} r(x,y)\, i(x,y)^2 \quad (\text{Thomson energy})$$

Def $R_{\mathrm{eff}}(u,v) := \inf \left\{ \tilde{\mathcal{E}}(i) : i \in \mathcal{I}_{u,v},\ \mathrm{val}(i) = 1 \right\}$.

$$\boxed{\textbf{Thm} \ \forall u,v \in V \text{ distinct}: \quad R_{\mathrm{eff}}(u,v) = \frac{1}{C_{\mathrm{eff}}(u,v)}}$$

Pf

$) = -i(y,x)$
$,y) = 0$ }

on energy)

$_{,v}$, $val(i)=1$ }

Pf Let $f(u)=1$, $f(v)=0$, $i \in \mathcal{I}_{u,v}$, $val(i)=1$

$$1 = (f(u)-f(v))\,val(i)$$

$$= \sum_{x\in V} f(x) \sum_{y:(x,y)\in E(V)} i(x,y)$$

$$= \sum_{(x,y)\in E(V)} f(x)\, i(x,y)$$

$$= \frac{1}{2}\sum_{(x,y)} (f(x)-f(y))\, i(x,y)\sqrt{c(x,y)}\sqrt{r(x,y)}$$

$$\overset{CS}{\leq} \mathcal{E}(f)^{1/2}\,\tilde{\mathcal{E}}(i)^{1/2}$$

$$\Rightarrow C_{eff}(u,v)\,R_{eff}(u,v) \geq 1$$

Now

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Now
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Now let f_* be as above, set

$$i(x,y) := c(x,y)\,[f_*(x) - f_*(y)] \longleftarrow \underline{\text{Ohm's law}}$$

<u>claim</u> $i \in \mathcal{I}_{u,v}$:

<u>Pf</u> $Pf_* = f_*$ on $V \setminus \{u,v\}$

$$\Rightarrow \forall x \neq u,v: \quad \sum_{y \in V} \underbrace{c(x,y)\,[f_*(x) - f_*(y)]}_{i(x,y)} = 0$$

<u>claim</u> $\mathrm{val}(i) = C_{\text{eff}}(u,v)$

<u>Pf</u>
$$\sum_y i(u,y) = \sum_y c(x,y)\,[f_*(u) - f_*(y)]$$
$$= \pi(u) \sum_y P(u,y)\,[f_*(u) - f_*(y)]$$
$$= \pi(u)\,[1 - Pf_*(u)] \overset{\text{above}}{=} C_{\text{eff}}(u,v).$$

Now
$$R_{\text{eff}}(u,v) \leq \tilde{\mathcal{E}}\left(\frac{i}{\mathrm{val}(i)}\right) = \frac{1}{\mathrm{val}(i)^2}\,\tilde{\mathcal{E}}(i)$$
$$= \frac{1}{\mathrm{val}(i)^2} \sum_{(x,y)} r(x,y)\,\big[c(x,y)\,(f_*(x) - f_*(y))\big]^2 = \frac{1}{\mathrm{val}(i)^2}\,\mathcal{E}(f_*)$$
$$= \frac{1}{C_{\text{eff}}(u,v)^2}\,C_{\text{eff}}(u,v) = \frac{1}{C_{\text{eff}}(u,v)} \quad \boxtimes$$

<u>Corollary</u> Given $\emptyset \neq A \subsetneq V$, let

$$G^A(x,y) := E^x\left(\sum_{k=0}^{\tau_{A^c}-1} 1_{X_k=y}\right)$$

Green function in A

Then

$$R_{eff}(u,v) = \frac{1}{\pi(u)} G^{\{v\}^c}(u,u)$$

<u>Pf</u>: skipped

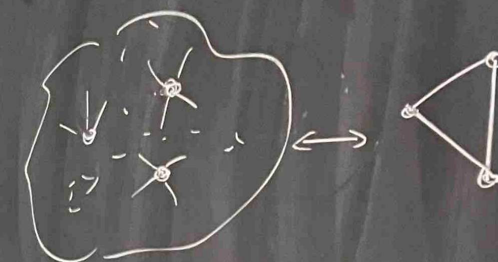
Network reduction

Lemma Let $\emptyset \neq U \subseteq V$. Then $\exists\, \tilde{c} : U \times U \to \mathbb{R}_+$, $\tilde{c}(x,y) = \tilde{c}(y,x)$ $x,y \in U$.
$(\neq C_{\mathrm{eff}}(x,y))$

s.t. $\forall g : U \to \mathbb{R}$:

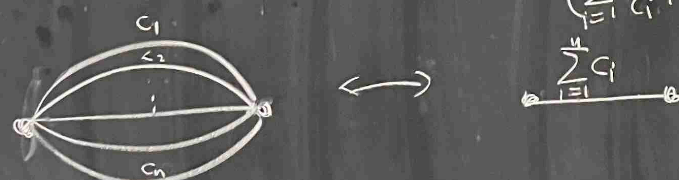
$$\inf_{f : V \to \mathbb{R}} \{ \mathcal{E}(f) : f|_U = g \} = \frac{1}{2} \sum_{x,y \in U} \tilde{c}(x,y) \left[g(y) - g(x) \right]^2$$

Pf: HW.



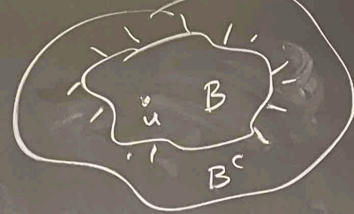
- series law

- parallel law

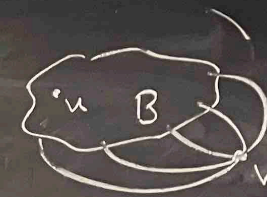


- star-triangle

shortening



$\longrightarrow$ $f|_{B^c} = 0$



Extension to ∞ - graphs

Let V be infinite, $u \in V$, $u \in B \subseteq V$ finite

$$C_{eff}(u, B^c) := \inf\left\{ \mathcal{E}(f) : \begin{array}{l} f|_{B^c} = 0 \\ f(u) = 1 \end{array} \right\}$$

For $R_{eff}(u, B^c)$ we define $\mathcal{I}_{u,B^c}$ = flows from u to B^c (set to zero outside B)

$$R_{eff}(u, B^c) := \inf\left\{ \tilde{\mathcal{E}}(i) : i \in \mathcal{I}_{u,B^c}, \ val(i) = 1 \right\}$$

Fact: $u \in B \subseteq \tilde{B}$: $C_{eff}(u, \bar{B}) \leq C_{eff}(u, B)$

$$R_{eff}(u, \bar{B}) \geq R_{eff}(u, B)$$

Def $R_{\text{eff}}(u,\infty) := \sup\limits_{\substack{B \subseteq V \\ u \in B}} R_{\text{eff}}(u, B^c)$

$C_{\text{eff}}(u,\infty) := \inf\limits_{\substack{B \subseteq V \\ u \in B}} C_{\text{eff}}(u, B^c)$

Lemma For $X = MC$ associated with electric network, $\forall u \in V$:

$$R_{\text{eff}}(u,\infty) \begin{cases} < \infty \\ = \infty \end{cases} \iff X \begin{cases} \text{transient} \\ \text{recurrent} \end{cases}$$

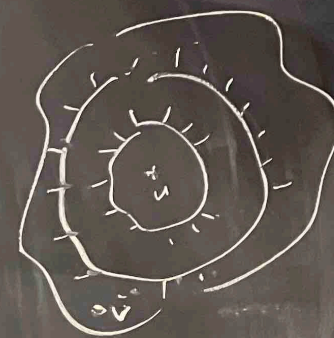
Pf $C_{\text{eff}}(u,B^c) = \pi(u)\, P^u(\tau_{B^c} < \hat{\tau}_u)$

Let $B_n \uparrow V$. Then $P^u(\tau_{B_n^c} \leq k) \xrightarrow{\forall k \geq 1} 0$ $\boxtimes$

So $P^u(\tau_{B_n} < \hat{\tau}_u) \longrightarrow P^u(\hat{\tau}_u = +\infty)$.

<u>Thm</u> (Nash-Williams' criterion)

Suppose $\exists\, \Pi = \{ E_k : k \geq 1\}$ are disjoint cutsets of u from v in $E(V)$. Then



$$R_{\text{eff}}(u,v) \geq \sum_{k\geq 1} \frac{1}{\sum_{(x,y)\in E_k} c(x,y)}$$

<u>Pf</u>: <u>Fact</u> $i_* \in I_{u,v}$, $\text{val}(i_*)=1$, E cutset of u from v.

Then $\sum_{e\in E} |i_*(e)| \geq 1$.

$$R_{\text{eff}}(u,v) = \mathcal{E}(i_*) \geq \sum_{k\geq 1} \sum_{(x,y)\in E_k} r(x,y)\, i_*(x,y)^2$$

$$\geq \sum_{k\geq 1} \frac{\sum_{(x,y)\in E_k} |i_*(x,y)|}{\sum_{(x,y)\in E_k} r(x,y)} \geq \sum_{k\geq 1} \frac{1}{\sum_{(x,y)\in E_k} r(x,y)}$$

$\boxtimes$

Corollary Given $\emptyset \neq A \subsetneq V$, let

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Green function in A

Then

$$R_{eff}(u,v) = \frac{1}{\pi(u)} G^{\{v\}^c}(u,u)$$

Pf: skipped