Continuing harmonic analysis

last time $h : \mathbb{Z}^d \to \mathbb{R}_+$ P-harmonic, $P = RW$
$\Rightarrow h$ is a combination of exponentials. $\exp\{t \cdot x\}$.

Corollary For all non-constant RW's with symmetric steps,
all ≥ 0 harmonic functions are constant.

Pf $$\sum_x Q(x) e^{t\cdot x} \underset{Q(x)=Q(-x)}{=} \sum_x Q(x)\left(\tfrac{1}{2}e^{t\cdot x} + \tfrac{1}{2}e^{-t\cdot x}\right) = \sum_x Q(x)\underbrace{\cosh(t\cdot x)}_{\geq 1}$$

Corollary Suppose step distribution obeys $\forall t \in \mathbb{R}^d$: $P(t\cdot X_1 > 0) > 0$. Then:
(1) if $X_1 \in L^1 \wedge EX_1 = 0 \Rightarrow$ All $h \geq 0$ harmonic functions are constant.
(2) if $\forall t \in \mathbb{R}^d$: $E(e^{t\cdot X}) < \infty \wedge EX_1 \neq 0 \Rightarrow \exists \geq 0$ non-constant harmonic function.

Pf $D := \{ t \in \mathbb{R}^d : E(e^{t\cdot X_1}) < \infty \}$,

Jensen's ineq. D convex.

So if $\exists t \neq 0$: $E(e^{tX_1}) = \sum_{x \in \mathbb{Z}^d} Q(x) e^{t\cdot x} = 1$

Then $s \mapsto \log E(e^{s\, t\cdot X_1})$ is convex on $s \in [0,1]$.

Since $t\cdot X_1$ non constant $\Rightarrow \frac{d}{ds} \log E(e^{s\, t\cdot X_1})\Big|_{s=0^+}$
$= E(t\cdot X_1) < 0$

(1) $EX_1 = 0 \Rightarrow$ such cannot exist.

(2) Suppose $E(t\cdot X_1) < 0$
but $P(t\cdot X_1 > 0) > 0 \Rightarrow E(e^{s\, t\cdot X_1}) > 1$ for $s \gg 1$.
$\Rightarrow \exists s_0 \in (0,\infty) : E(e^{s\, t\cdot X_1}) = 1$ $\boxtimes$

$)e^{t.x}=1$

$s\in[0,1].$

$t.x)\Big|_{s=0^+}$

$X_1)<0$

$X_1)>1$

$>1.$ $\boxtimes$

Q: What happens for Markov chains?

Def Let $\Lambda\subseteq S$. We say $h: S\to\mathbb{R}$ is P-harmonic in Λ if $\forall x\in\Lambda$: $P|h|(x)<\infty$ $\wedge$ $Ph(x)=h(x)$

Thm (Dirichlet problem) Let $\Lambda\subseteq S$ and denote $\tau_{\Lambda^c}:=\inf\{n\geq 0: X_n\in\Lambda^c\}$.
Assume $\forall x\in\Lambda$: $P^x(\tau_{\Lambda^c}<\infty)=1$. Given $g\in\ell^\infty(S)$ set

$$h(x):=E^x\left(g(X_{\tau_{\Lambda^c}})\right)$$

Then ...

$Ph=h\Rightarrow$
$$\sum_{y\in S}P(x,y)\,[h(y)-h(x)]=0$$

h s

This

Pf: [

h uni

M

OS

h solves Dirichlet problem

$$\begin{cases} Ph(x) = h(x) & x \in \Lambda \\ Ph(x) = g(x) & x \notin \Lambda \end{cases} \qquad (*)$$

This solution is unique in $\ell^\infty(S)$.

<u>Pf</u>: $\boxed{h \text{ is a solution}}$

Let $x \in \Lambda$. Then $P^x(\tau_{\Lambda^c} \geq 1) = 1$. M.P:

$$h(x) = \sum_y E^x\left(1_{X_1 = y}\, g(X_{\tau_{\Lambda^c}})\right) = \sum_y P(x,y)\, \underbrace{E^y\left(g(X_{\tau_{\Lambda^c}})\right)}_{= h(y)}$$

$\boxed{h \text{ unique}}$ Let f be bounded solution. Then

$M_n = f(X_{n \wedge \tau_{\Lambda^c}})$ is a martingale, bounded

OST: $\forall x \in S$: $f(x) = E^x(M_0) = E^x(M_{\tau_{\Lambda^c}}) = E\left(g(X_{\tau_{\Lambda^c}})\right)$

$\boxtimes$

Q: $\Lambda^c := \{z\}$. What is $P^x(\tau_{\{z\}} < \infty) =: \tilde{h}(x)$

$\tilde{h}(z) = 1$, $\quad P\tilde{h}(z) = \sum_{x \in S} P(z,x)\, P^x(\tau_{\{z\}} < \infty)$

$T_z = \inf\{n \geq 1 : X_n = z\}$.

$= P^z(T_z < \infty)$

$\tilde{h}(z) - P\tilde{h}(z) = 1 - P^z(T_z < \infty) = P^z(T_z = \infty) > 0$

$\Leftrightarrow$ z transient.

So $P^x(\tau_{\Lambda^c} = +\infty) = 0$ for recurrent chains

and $\neq 0$ for transient chains if Λ^c "small"

Def A function $f : S \to \mathbb{R}$ is

<u>superharmonic</u> at x if $Pf(x) \leq f(x)$ $\quad \left(P|f|(x) < \infty \right)$

<u>subharmonic</u> at x if $Pf(x) \geq f(x)$

Thm An irreducible MC on S is transient if and only if there exists non-constant a nonnegative (bonded), superharmonic function

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Pf $\Longrightarrow h(x) = P^x(T_x < \infty)$ is bdd, ≥ 0, superharmonic

noncontant if chain transient.

$\Longleftarrow$ h superharmonic, ≥ 0 ; $M_n = h(X_n)$ supermartingale

$M_\infty := \lim_{n \to \infty} M_n$ exists a.s.

recurrence $\Rightarrow$ $\forall x \in S$: $h(x) = M_\infty$. $\boxtimes$

Thm (Riesz decomposition) Assume P transient.

Let $f : S \to \mathbb{R}_+$ be superharmonic. Then

(1) $h(x) := \lim_{n \to \infty} P^n f(x)$ exists $\forall x \in S$.
and is harmonic

(2) Setting $g(x) := f(x) - Pf(x) \geq 0$.
we have

$$\forall x \in S: \quad f(x) = \overbrace{\sum_{y \in S} G(x,y) g(y)}^{\text{potential}} + h(x)$$

where $G(x,y) := \sum_{n \geq 0} P^n(x,y)$ is Green function.

Pf :

Pf: $f(x) = \sum_{k=0}^{n-1} \left[P^k f(x) - P^{k+1} f(x) \right] + P^n f(x)$

$= \sum_y \left(\sum_{k=0}^{n-1} P^k(x,y) \right) g(y) + P^n f(x).$

Limit exists by MCT. NTS: $h = Ph$.

Note

$P^{n+1} f(x) = \sum_{y \in S} P(x,y) \underbrace{P^n f(y)}_{\leq f(y) \text{ by superharmonicity}}$

$\xrightarrow{DCT} \sum_{y \in S} P(x,y) h(y) \qquad \boxtimes$

Note $x \longmapsto G(x,y)$ solves Poisson eq

$$G(x,y) = \delta_{xy} + \sum_{z \in S} P(x,z)\, G(z,y)$$

So $$PG(\cdot\,,y)(x) - G(x,y) = -\delta_{x,y}.$$

$$G = (\mathrm{id} - P)^{-1} \quad (\text{Need } \ell^2\text{-space})$$